Hyperdoctrine semantics for higher-order modal logic

Florrie Verity

Australian National University

florrie.verity@anu.edu.au

YaMCATS 27
May 12, 2022

Joint work with Yoshihiro Maruyama

Algebraic semantics for modal logic

Propositional modal logic has nice algebraic semantics.

- Intuitionistic logic $\leftrightarrow$ Heyting algebras
- Classical logic $\leftrightarrow$ Boolean algebras
- Modal logic $\leftrightarrow$ "Modal algebra" (Boolean algebra + operator)

$$(A, \wedge_A, \vee_A, \neg_A, \top_A, \bot_A), \quad \Box_A : A \to A, \quad \Diamond_A = \neg_A \Box_A \neg_A$$

and zero or more conditions on $\Box_A$, e.g.:

Axiom	**Condition on $\Box$**
M: $\Box(\phi \wedge \psi) \supset (\Box\phi \wedge \Box\psi)$	$\Box_A(x \wedge_A y) \leq (\Box_A x \wedge_A \Box_A y)$
C: $(\Box\phi \wedge \Box\psi) \supset \Box(\phi \wedge \psi)$	$(\Box_A x \wedge_A \Box_A y) \leq \Box_A(x \wedge_A y)$
N: $\Box\top = \top$	$\Box_A \top_A = \top_A$

Semantics for quantified modal logic

Traditional approach:

- Many complete propositional modal logics have incomplete extensions w.r.t. traditional Kripke
- …and these aren't just "cooked up"

Categorical approach:

- Ghilardi's hyperdoctrine semantics for first-order modal logic (K and stronger) [1]
- Awodey, Kishida and Kotzsch's algebraic topos semantics for higher-order intuitionistic S4 modal logic [2]

[1] Torben Braüner & Silvio Ghilardi (2007): First-order modal logic. Studies in Logic and Practical Reasoning 3, pp. 549–620.

[2] Steve Awodey, Kohei Kishida & Hans-Christoph Kotzsch (2014): Topos Semantics for Higher-Order Modal Logic. Logique et Analyse 57(228), pp. 591–636.

Our work

We've extended the hyperdoctrine semantics in two ways:

- for weaker "non-normal" modal logics;
- for higher-order modal logic;

and proven soundness and completeness.

Lawvere hyperdoctrines

- Originally devised in [3] for intuitionistic predicate logic – but are flexible
- Account for quantifiers via adjoints
- Reduce to standard algebraic semantics on the propositional level
- “Logic over type theory” perspective
  - e.g. formulae are given with type contexts: e.g. $\phi\ [\Gamma]$, for $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$

[3] F. William Lawvere (1969): Adjointness in Foundations. Dialectica 23, pp. 281–296.

Lawvere hyperdoctrines

For $\mathbf{C}$ with finite products, a Lawvere hyperdoctrine is a functor

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA},$$

such that for every projection $\pi : X \times Y \to Y$ in $\mathbf{C}$,
$P(\pi) : P(Y) \to P(X \times Y)$ has right and left adjoints, denoted

$$\forall_\pi, \exists_\pi : P(X \times Y) \to P(Y),$$

that satisfy corresponding Beck-Chevalley conditions:

$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{\forall_\pi} & P(Y) \\ {\scriptstyle P(\mathrm{id}_X \times f)} \downarrow & & \downarrow {\scriptstyle P(f)} \\ P(X \times Z) & \xrightarrow[\forall_{\pi'}]{} & P(Z) \end{array}$$

(and $\exists_\pi$ satisfies Frobenius reciprocity).

Hyperdoctrine semantics

Syntax	Semantics
types σ	$[\![\sigma]\!] \in \mathrm{obj}(\mathbf{C})$
function symbols $F : \sigma_1, \ldots, \sigma_n \to \tau$	$[\![F]\!] : [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!] \in \mathrm{ar}(\mathbf{C})$
predicate symbols $R\ [\Gamma]$ (for $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$)	$[\![R\ [\Gamma]]\!] \in P([\![\Gamma]\!])$
terms $t : \tau\ [\Gamma]$	inductively on structure of t
formulae $\phi\ [\Gamma]$	inductively on structure of ϕ

A formula $\phi\ [\Gamma]$ is *satisfied* in an interpration $[\![\text{-}]\!]$ in a hyperdoctrine P if and only if

$$[\![\phi]\!] = \top_{P([\![\Gamma]\!])}.$$

Modal hyperdoctrine

For $\mathbf{C}$ a category with finite products, a modal hyperdoctrine is a functor

$$P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA},$$

where $\mathbf{MA}$ is the category of modal algebras and their homomorphisms (and P satisfies the aforementioned conditions for quantifers).

- Modal formulae have the interpretation:

$$[\![\Box\phi\ [\Gamma]]\!] := \Box_{P([\![\Gamma]\!])}([\![\phi\ [\Gamma]]\!]).$$

- We have to specify in our syntax that $\Box$ commutes with substitution.

For *non-normal* modal logics, we just take fewer conditions on the modal algebra operator.

Higher-order hyperdoctrines

A higher-order hyperdoctrine (aka tripos) is a hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ such that:

- the base category $\mathbf{C}$ is a cartesian closed category;
- there is an object Ω in $\mathbf{C}$ such that there is an isomorphism

$$P(C) \simeq \mathrm{Hom}_{\mathbf{C}}(C, \Omega)$$

  natural in C.

Correspond to toposes via two functors:

- taking subobject hyperdoctrines;
- the tripos-to-topos construction.

Higher-order syntax

We make two adjustments:

- add arrow and finite product types to the underlying type theory;
- add a distinguished type Prop to the type signature, to reflect the logical structure into the type structure.
  - For each relation symbol $R \subseteq \sigma_1, \ldots, \sigma_n$ in the signature, introduce a corresponding function symbol $R : \sigma_1, \ldots, \sigma_n \to \mathsf{Prop}$.
  - Add a rule to relate logical equivalence between formulae to equality of terms of type Prop:

$$\frac{\vdash_{\mathbf{HoS}} \phi \supset\subset \psi \; [\Gamma]}{\phi = \psi : \mathsf{Prop} \; [\Gamma]} \; (\textit{Prop})$$

Logical meaning of the isomorphism in the previous definition:

$$P(\Gamma) \simeq \mathrm{Hom}_{\mathbf{C}}(\Gamma, \mathsf{Prop})$$

Higher-order modal hyperdoctrines

A higher-order **modal** hyperdoctrine (aka **modal** tripos) is a **modal** hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MA}$ such that:

- the base category $\mathbf{C}$ is a cartesian closed category;
- there is an object Ω in $\mathbf{C}$ such that there is an isomorphism

$$P(C) \simeq \mathsf{Hom}_{\mathbf{C}}(C, \Omega)$$

natural in C.

Conclusion

- The traditional approach for modal logic semantics doesn't extend well to quantified modal logic,
- but the categorical approach of hyperdoctrine semantics works very nicely, in both the first-order and higher-order cases and for very weak modal logics.