Specialised session 3; Pseudofinite groups

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Part 1

Simple pseudofinite groups

Ultraproducts, a reminder from Lecture 3

A subset $U \subseteq P(I)$ s.t. $I \in U$, $\emptyset \notin U$, U is closed under finite intersections and supersets and U is maximal w.r.t this is an *ultrafilter* on I. If $U = \{X \subseteq I : i \in X\}$ for some $i \in I$, then U is principal.

- Let $\{M_i : i \in I\}$ be a family of L-structures (for fixed L), U a non-principal ultrafilter on I and $M^* := \prod_{i \in I} M_i$.
  - A property P holds *for almost all* i if $\{i : P \text{ holds for } M_i\} \in U$.
  - Define an equivalence relation $\sim$ on M^* as: $(x_i) \sim (y_i) \text{ if and only if } \{i \in I : x_i = y_i\} \in U$.
  - Relations functions and constants of L are defined to hold of a tuple of $M = M^*/\sim$ if they hold for almost all i.
  - The resulting L-structure $M = \prod_{i \in I} M_i/U$ is called the *ultraproduct* of the L-structures M_i w.r.t the ultrafilter U.

Theorem 1.1 (Łoś's Theorem). Let $M = \prod_{i \in I} M_i/U$ be as above, $\phi(\overline{x})$ be an L-formula and $\overline{[(a_i)]} \in M$. Then

$$\prod_{i \in I} M_i/U \models \phi(\overline{[(a_i)]}) \Leftrightarrow \{i \in I : M_i \models \phi(\overline{a_i})\} \in U.$$

Pseudofinite structures

Fix a language L.

Definition 1.2. An infinite L-structure M is called pseudofinite if for every L-sentence σ s.t. $M \models \sigma$ there is a finite L-structure M_0 s.t. $M_0 \models \sigma$.

Let FIN be the common theory of all finite L-structures.

Proposition 1.3. Let M be an infinite L-structure. Then TFAE:

1. *M is pseudofinite.*
2. *$M \equiv \prod_{i\in I} M_i/U$ where M_i's are finite and U is a non-principal ultrafilter on I.*
3. *$M \models \mathrm{FIN}$.*

Proof.
- $(2) \Rightarrow (3)$: Let $M \equiv \prod_{i\in I} M_i/U$. Then $\sigma \in \mathrm{FIN} \Rightarrow M_i \models \sigma$ for any L-sentence σ. Thus, $\{i \in I : M_i \models \sigma\} \in U$ and hence by Łos's Theorem $M \models \mathrm{FIN}$.
- $(3) \Rightarrow (1)$: Let $M \models \mathrm{FIN}$ and consider an L-sentence σ s.t. $M \models \sigma$. If σ has no finite models then, for any finite L-structure M_0, $M_0 \models \neg\sigma$. It follows that $M \models \neg\sigma$.
- $(1) \Rightarrow (2)$: Let I be the collection of finite subsets of $\mathrm{Th}(M)$. For $i \in I$, let M_i be a finite L-structure s.t. $M_i \models i$. For all i, let $X_j = \{j \in I : M_j \models \phi \text{ for all } \phi \in i\}$; the collection F of X_j's has *fip* (check: $X_i \cap X_j = X_{i\cup j} \neq \emptyset$) and so we may extend F to a non-principal ultrafilter U. If $M \models \sigma$ then $\{i \in I : M_i \models \sigma\} \supseteq X_\sigma \in U$, and so, by Łos's Theorem, $\prod_{i\in I} M_i/U \models \sigma$.

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Proof. ■ $(2) \Rightarrow (3)$: Let $M \equiv \prod_{i \in I} M_i / U$. Then $\sigma \in \mathrm{FIN} \Rightarrow M_i \models \sigma$ for any L-sentence σ. Thus, $\{i \in I : M_i \models \sigma\} \in U$ and hence by Łos's Theorem $M \models \mathrm{FIN}$.

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Pseudofinite fields

Theorem 1.4 (Ax 1968). *An infinite field F is pseudofinite iff all of the following hold*

1. *F is perfect, i.e., $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) = p$ and every element has a p^{th} root.*
2. *F is quasi-finite, i.e., inside a fixed algebraic closure, F has a unique extension of each finite degree;*
3. *F is pseudo-algebraically closed (PAC), i.e., every absolutely irreducible variety which is defined over F has a F-rational point.*

Examples 1.5.

- In $\mathrm{char}(F) = 0$: $\prod_{p \in P} \mathbb{F}_p / U$ where P is the set of all prime numbers and U is a non-principal ultrafilter on P.
- In $\mathrm{char}(F) > 0$: An infinite subfield F of $\mathbb{F}_p^{alg}$ generated by $\mathbb{F}_{p^2}, \mathbb{F}_{p^3}, \mathbb{F}_{p^5}, \ldots$.

Example 1.6. Non-examples of pseudofinite fields include:

- An algebraically closed field K: If $\mathrm{char}(K) \neq 2$ consider $\sigma := \forall x \exists y (y^2 = x)$; show that if K was pseudofinite then $1 = -1$. If $\mathrm{char}(K) = 2$ consider $\sigma_2 := \forall x \exists y (y^3 = x)$; show that if K was pseudofinite then $1 = 0$.
- Stable fields.

Pseudofinite groups, examples and non-examples

Examples 1.7. The following groups are *not* pseudofinite.

- $(\mathbb{Z},+)$: the FO-expressible statement 'If the map sending $x \mapsto 2x$ is injective then it is surjective' which is true in all finite groups is false in $(\mathbb{Z},+)$.
- $(\mathbb{Z}_p,+)$ and $(\mathbb{Z}_{p^\infty},+)$: consider the map $x \mapsto px$.
- Free groups F_n: contains a definable subgroup isomorphic to $(\mathbb{Z},+)$.
- A (twisted) Chevalley group $X(K)$ over a non-pseudofinite field K, e.g., $\mathrm{PSL}_2(K)$ for K algebraically closed.

Examples 1.8. Examples of pseudofinite groups include:

- Infinite definable subgroups of pseudofinite groups and infinite quotients of a pseudofinite group by a definable normal subgroup.
- Infinite extraspecial p-groups of odd exponent p.
- $(\mathbb{Q},+) \equiv \prod_{p\in P} C_p/U$, where U is a non-principal ultrafilter on the set P of all prime numbers and C_p is the cyclic group of order p.
- A (twisted) Chevalley group $X(F)$ over a pseudofinite field, e.g., $\mathrm{PSL}_2(F)$ for F pseudofinite.

Q (Sabbagh): Are there any finitely generated pseudofinite groups?

CFSG

Theorem 1.9 (CFSG). Every finite simple group is isomorphic to one of the following groups.

- *A cyclic group of prime order.*
- *An alternating group Alt_n, for $n \geqslant 5$.*
- *Simple group of Lie type over some finite field.*
- *One of the 26 sporadic groups.*

Definition 1.10. Let G be a group. The socle of G, denoted by $\mathrm{Soc}(G)$, is the subgroup of G generated by all minimal normal subgroups.

Note: the socle of a finite group is a direct product of simple groups.

Simple pseudofinite groups

Theorem 1.11 (Wilson 1995). *G is a simple pseudofinite group iff $G \equiv X(F)$, where $X(F)$ is a (twisted) Chevalley group over a pseudofinite field F.*

Proof of $\Leftarrow$:

1. (Point) Let $\{X(F_i) : i \in I\}$ be a family of finite or pseudofinite (twisted) Chevalley groups of the fixed Lie type X and U bea non-principal ultrafilter on I. Then $\prod_{i \in I} X(F_i)/U \cong X(\prod_{i \in I} F_i/U)$.
2. (Keisler-Shelah) Two L-structures M and N are elementarily equivalent if and only if they have isomorphic ultrapowers.

$X(F)$ is pseudofinite: let $F \equiv F_1$, where F_1 is an ultraproduct of finite fields. Let F^* and F_1^* be ultrapowers of F and F_1 resp. Since $F_1^* = \prod_{i \in I} \mathbb{F}_i/U$ for $\mathbb{F}_i$ a finite field, by (1) and (2) we have:

$$X(F) \equiv X(F)^* \cong X(F^*) \cong X(F_1^*) \cong \prod_{i \in I} X(\mathbb{F}_i)/U.$$

Idea of the proof of $\Rightarrow$ of Theorem 1.11

Step 1: Find an L-sentence σ s.t.

1. For any non-abelian simple group G, $G \models \sigma$.
2. Any finite group G satisfying σ has a non-abelian simple socle $\mathrm{Soc}(G)$.

$$\sigma := \forall x, y \left[(x \neq 1 \wedge C_G(x,y) \neq 1) \Rightarrow \left(\bigcap_{g \in G} (C_G(x,y) C_G(C_G(x,y)))^g = 1 \right) \right].$$

Proof of (1): Let G be as in (1) and $x, y \in G$ s.t. $x \neq 1$ and $C_G(x,y) = H \neq 1$. Then

$$\bigcap_{g \in G} (HC_G(H))^g \trianglelefteq G \text{ so either } \bigcap_{g \in G} (HC_G(H))^g = 1 \text{ or } \bigcap_{g \in G} (HC_G(H))^g = G.$$

In the latter case $H = G$ which is a contradiction as $Z(G) = 1$. □

Step 2: Every simple pseudofinite group is $\equiv$ to an ultraproduct of finite simple groups.

- These finite groups are of a given class; alternating groups may be easily eliminated:

Exercise 1.12. Let $G = \prod_{i \in I} Alt_{n_i}/U$ be an infinite group s.t $n_i \geqslant 5$ and U is a non-principal ultrafilter on I. Let $(x_i) \in \prod_{i \in I} Alt_{n_i}$ s.t. $x_i = (12)(34)$ for all i. Then the group generated by the conjugacy class of $[(x_i)] \in G$ is a proper normal subgroup of G.

- One may prove that G is an ultraproduct of Chevalley groups of the same Lie type X with fixed Lie rank n over finite fields of increasing order.

Some comments

- (Ryten) In Theorem 1.11, '$\equiv$' may be replaced with '$\cong$'.

Recall that a group G is said to be of centraliser dimension k, denoted by $\mathrm{cd}(G) = k$, if the longest proper descending chain of centralisers in G has length $k \in \mathbb{N}$.

- (Uğurlu) A definably simple (no proper non-trivial definable normal subgroups) pseudofinite group G of finite centraliser dimension is simple.

Part 2

Stable pseudofinite groups

The radical

The solvable radical $Rad(G)$ of a group G, if exists, is the maximal normal solvable subgroup of G. Note that $Rad(G) \trianglelefteq G$.

Theorem 2.1 (Wilson 2006/2009). —

1. *There is an L_{group}-sentence σ_{56} s.t. if G is a finite group then*

$$G \models \sigma_{56} \Leftrightarrow G \text{ is solvable.}$$

2. *There is an L_{group}-formula $\phi_R(x)$ s.t. if G is a finite group then*

$$Rad(G) = \{g \in G : G \models \phi_R(g)\}.$$

Note: it is *not* true that a pseudofinite group G is solvable iff $G \models \sigma_{56}$—if G is an ultraproduct of finite solvable groups G_i without a common bound on their derived lengths then $G_i \models \sigma_{56}$ and thus by Łos's Theorem $G \models \sigma_{56}$.

- Let G be a pseudofinite group and set $R(G) = \phi_R(G)$. We call $R(G)$ the radical of G. As $R(G) \trianglelefteq G$ is definable, $\overline{G} = G/R(G)$ is either a pseudofinite group or a finite group. Further $\overline{G}$ semi-simple, i.e., has no non-trivial abelian normal subgroups; note that semi-simplicity is a FO-expressible property by a sentence $(\forall x \neq 1)(\exists y)([x, x^y] \neq 1)$. Therefore, ϕ_R allows us to split G into a semi-simple part $G/R(G)$ and a 'solvable-like' part $R(G)$.

Stability and generalisations; stable groups

Definition 2.2. A formula $\phi(\overline{x}, \overline{y})$ is unstable (w.r.t T) if there are $\overline{a}_i \in M^n$, $\overline{b}_i \in M^m$ ($i \in \omega$) and $M \models T$ such that for all $i, j \in \omega$, $M \models \phi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i < j$.

Definition 2.3. A formula $\phi(\overline{x}, \overline{y})$ has IP (w.r.t T) if there are $M \models T$ and $\overline{a}_i \in M^n$ s.t. for all $S \subset \omega$ there is $\overline{b}_S \in M^m$ with, for all $i \in \omega$ $M \models \phi(\overline{a}_i, \overline{b}_S) \Leftrightarrow i \in S$.

Chain conditions in stable groups

Definition 2.4. A family $\{H_i : i \in I\}$ of subgroups of a group G is called uniformly definable if for some $\phi(x, y)$ we have $H_i = \phi(M, a_i)$ for some parameter a_i, for all $i \in I$.

Trivial chain condition. Let G be a stable group. For any formula $\phi(x, y)$ there is $n_\phi \in \mathbb{N}$ s.t. every chain $H_1 \subseteq H_2 \subseteq \ldots$ of subgroups of G uniformly defined by ϕ has length at most n_ϕ.

Proposition 2.5. Let G be an NIP group. For any formula $\psi(x, y)$ there is $n_\psi \in \mathbb{N}$ s.t. if I is finite and $\{H_i : i \in I\}$ is a family of subgroups of G uniformly definable by ψ then $\bigcap_{i \in I} H_i = \bigcap_{i \in I_0} H_i$ for some $I_0 \subseteq I$ with $|I_0| \leqslant n_\psi$.

Exercise 2.6. Prove Proposition 2.5 via contradiction.

Combining Trivial chain condition and Proposition 2.5 we get:

Theorem 2.7 (Baldwin and Saxl 1976). Let G be a stable group. Then for any formula $\phi(x, y)$ there is $k_\phi \in \mathbb{N}$ s.t any descending chain of intersections of ϕ-definable subgroups has length at most k_ϕ.

- Applying Theorem 2.7 to the formula $\phi(x, y)$ saying $xy = yx$, one observes that stable groups have descending chain condition on centralisers.

$R(G)$ of a stable group G

Theorem 2.8 (Khukhro 2009). Let G be a group with $\mathrm{cd}(G) = k$ s.t. one of the following hold

1. *G is locally solvable.*
2. *G is elementary equivalent to an ultraproduct of finite solvable groups.*

Then G is solvable of k-bounded derived length.

Note: by Theorem 2.8, in a locally finite group G with $\mathrm{cd}(G) = k$, then $\mathrm{Rad}(G)$ exists.

Proposition 2.9. Let G be a pseudofinite group with $\mathrm{cd}(G) = k$. Then $R(G)$ is solvable of k-bounded derived length.

Proof. We have $\mathrm{cd}(R(G)) \leqslant k$ and $R(G) \equiv \prod_{i\in I} \mathrm{Rad}(G_i)/U$. By Łos's Theorem $\mathrm{cd}(\mathrm{Rad}(G_i)) \leqslant k$, for almost all i. Therefore by Theorem 2.8, $\mathrm{Rad}(G_i)$ is solvable of k-bounded derived length, for almost all i. It follows that $R(G)$ is solvable of k-bounded derived length.

□

- In particular, if G is a stable pseudofinite group then $R(G)$ is solvable.

Stable pseudofinite groups

Theorem 2.10 (MacPherson and Tent 2007). Let $G \equiv \prod_{i \in I} G_i/U$ be a stable pseudofinite group. Then there is r s.t. $[G : R(G)] \leqslant r$, that is, G is solvable-by-finite.

Proof. Denote $\overline{G} = G/R(G)$ and $\overline{G}_i = G_i/\mathrm{Rad}(G_i)$. We know that, for all i, $\mathrm{Soc}(\overline{G}_i) = \overline{S}_i = \overline{T}_{1,i} \times \cdots \times \overline{T}_{n,i}$ is a direct product of non-abelian simple groups.

- *Step 1: There is $t \in \mathbb{N}$ s.t. $n \leqslant t$ for almost all i.*

Assume contrary. Then, for $j = 1, \ldots, t$, we may pick $x_{j,i} \in \overline{T}_{j,i}$ so that

$$C_{\overline{S}_i}(x_{i,1}) > C_{\overline{S}_i}(x_{i,1}, x_{i,2}) > \ldots > C_{\overline{S}_i}(x_{i,1}, x_{i,2}, \ldots, x_{i,t})$$

is of length t. Thus, $\overline{G}$ is not of finite centraliser dimension contradicting stability. □

- *Step 2: There is $e \in \mathbb{N}$ s.t., for almost all i, if $\overline{T}_{j,i} \cong \mathrm{Alt}_n$, then $n \leqslant e$, and if $\overline{T}_{j,i} \cong X(F)$, then the Lie rank n of $X(F)$ is at most e.*

Alt_n has a proper chain of centralisers

$$C_{\mathrm{Alt}_n}(1) > C_{\mathrm{Alt}_n}((12)(34)) > \cdots > C_{\mathrm{Alt}_n}((12)(34), \ldots, (k-3, k-1)(k-1, k)),$$

where $k = n$ or $k = n - 1$. The length of such chain is $\lfloor \frac{n}{4} \rfloor$. Therefore, in the case $\overline{T}_{j,i} \cong \mathrm{Alt}_n$, there must be bound e on n, as $\overline{G}_i$ is of finite centraliser dimension. Similarly for $X(F)$'s: in a group of Lie type X, there are elements whose centralisers contain a group of Lie type X of lower Lie rank and so one may construct a proper descending chain of centralisers whose length increases when the Lie rank increases.

Stable pseudofinite groups cont.

- At this point we know that each, for almost all i,

$$\mathrm{Soc}(G_i/\mathrm{Rad}(G_i)) = \mathrm{Soc}(\overline{G}_i) = \overline{S}_i = \overline{T}_{1,i} \times \cdots \times \overline{T}_{t,i},$$

  where each $T_{j,i}$ is either a sporadic group, an alternating group of bounded size or a (twisted) Chevalley group of bounded Lie rank, for $j = 1, \ldots, t$.

- To finish the proof, we show that, for almost all i, there is $\ell \in \mathbb{N}$ s.t. $|\overline{S}_i| \leqslant \ell$. Then one obtains an upper bound on $|\mathrm{Aut}(\overline{S}_i)|$, and hence on $|\overline{G}_i|$ as $\overline{G}_i \leqslant \mathrm{Aut}(\overline{S}_i)$. Thus, $\overline{G} = G/R(G)$ is finite as claimed.
  - Assume that no such bound exists, that is, for all $\ell \in \mathbb{N}$, $|\overline{S}_i| > \ell$ for almost all i. Then, for almost all i, we may find (twisted) Chevalley subgroups of $\overline{T}_{j,i}$ of fixed Lie type n and fixed Lie rank m over arbitrarily large finite fields. Such groups are known to be uniformly simple and thus they are uniformly definable in $\overline{S}_i$. But then, the stable group $\overline{G}$ contains a definable (twisted) Chevalley group $X(F)$ over a pseudofinite field—this contradicts stability.

Remark. Stability is used in the proof of Theorem 2.10 in exactly two places:

1. To observe that $G/R(G)$, and thus $G_i/\mathrm{Rad}(G_i)$'s (for almost all i), are of finite centraliser dimension.
2. To observe that $|\overline{S}_i|$ has an upper bound, for almost all i.

Some more general results

Theorem 2.11 (MacPherson and Tent 2013). *Let $G \equiv \prod_{i \in I} G_i/U$ be an NIP pseudofinite group. Then there is d s.t. $[G : R(G)] \leqslant d$.*

Note: Theorem 2.12 does *not* imply that a pseudofinite *NIP* group is solvable-by-finite. However, if we add the assumption of centraliser chains, then this is the case:

Theorem 2.12 (MacPherson and Tent 2013). *Let $G \equiv \prod_{i \in I} G_i/U$ be an NIP pseudofinite group of finite centraliser dimension. Then there is d s.t. $[G : R(G)] \leqslant d$, that is, G is solvable-by-finite.*

Lemma 2.13. *Let G be a locally finite group with $\mathrm{cd}(G) = k$. Denote $\overline{G} = G/\mathrm{Rad}(G)$. Then the centraliser dimension of $\overline{G}$ is k-bounded.*

Combining the proof of Theorem 2.10 and Lemma 2.13 we get:

Theorem 2.14. *Let $G \equiv \prod_{i \in I} G_i/U$ be a pseudofinite group with $\mathrm{cd}(G) = k$. Then the radical $R(G)$ is a solvable group of k-bounded derived length and $\prod_{i \in I} \mathrm{Soc}(G_i/\mathrm{Rad}(G_i))/U = \overline{L}_1 \times \cdots \times \overline{L}_t$ is a finite direct product of simple non-abelian groups $\overline{L}_j$, each of which is either a finite group or a (twisted) Chevalley group over a pseudofinite field.*

Exercises 2.15. Let $M = \prod_{i \in I} M_i / U$ be an ultraproduct of L-structures M_i over a non-principal ultrafilter U on $I = \omega$. Prove that M is $\aleph_1$-saturated.

1. Let M be a pseudofinite structure and $f : M^k \Rightarrow M^k$ be a definable function. Prove that f is injective if and only if f is surjective.
2. A complete theory L-theory T is *pseudofinite* if every L-sentence σ s.t $T \models \sigma$ has a finite model. Give an example of a complete theory which
   1. is pseudofinite.
   2. is not pseudofinite.
3. Let G be a pseudofinite group. Prove that any infinite definable subgroup of G and any infinite quotient of G by a definable normal subgroup are pseudofinite.
4. Let A be an infinite abelian group. Prove that TFAE:
   1. A is definably simple.
   2. A is torsion-free divisible.
   3. $A \equiv (\mathbb{Q}, +)$.
5. Let G be a stable group and let $A \leqslant G$ be an abelian subgroup (not necessarily definable). (A) Prove that there is a definable abelian subgroup $A' \geqslant A$ of G. (B) Prove that the statement above is also true for nilpotent and solvable subgroups (Hint: use induction on nilpotency/solvability class).
6. A group G is called *bounded* if the derived lengths of its solvable normal subgroups are bounded. Let $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ be a pseudofinite group of centraliser dimension k. Prove that then G is bounded.

Exercises 2.15. ■ Let $M = \prod_{i \in I} M_i / U$ be an ultraproduct of L-structures M_i over a non-principal ultrafilter U on $I = \omega$. Prove that M is $\aleph_1$-saturated.

1. Let M be a pseudofinite structure and $f : M^k \Rightarrow M^k$ be a definable function. Prove that f is injective if and only if f is surjective.
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6. A group G is called *bounded* if the derived lengths of its solvable normal subgroups are bounded. Let $G \equiv \prod_{i \in I} G_i / \mathcal{U}$ be a pseudofinite group of centraliser dimension k. Prove that then G is bounded.