An excursion into Model Theory and its applications; Automorphism groups of countable first order structures

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LMS online lecture series – Fall 2020

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Lecture 1

Automorphism groups of first order structure

Notation and some group theory

- Assume G is a group acting (from left $G \times \Omega \to \Omega$) on Ω. For $a \in \Omega$ denote $Ga := \{ga : g \in G\}$ for the orbit of a under G. Orbits are equivalence classes.
- G is called transistive if it has only one orbit.
- If $A \subset \Omega$ denote G_A for $\{g \in G : ga = a, \forall a \in A\}$, the pointwise stabiliser of A, and $G_{\{A\}}$ for $\{g \in G : gA = A\}$ the setwise stablisier.
- For $n \in \mathbb{N}$ there is a natural action of G on Ω^n via diagonal action $g\bar{a} = (ga_1, \cdots, ga_n)$.
- A topological group G is a group with a topology on G such that the group's operation and the inverse function are continuous.
- Every group topology on G is determined by its neighbourhood at identity 1_G (follows from the fact that for every h the map $g \mapsto gh$ is a homeomorphism).

Some topology

- Let Ω be an infinite set and consider discrete topology on Ω. Consider the product topology on Ω^{Ω} i.e. the coarsest topology for which all the canonical projections are continuous. This is referred to τ_{tp} as the topology of pointwise convergence.
- Sets $O_f =: \{f' \in \Omega^{\Omega} : f' \text{ extends } f\}$, where finite map $f : \Omega \to \Omega$ form a base for the topology τ_{pt}.
- Let $Sym(\Omega)$ be the set of all permutations of Ω.
- $Sym(\Omega)$ is a topological group with the induced subspace topology on the product topology on Ω^{Ω} (Exercise).
- Basic open sets are closed hence $Sym(\Omega)$ is totally disconnected.
- When Ω is countable $Sym(\Omega)$ is separable (i.e. there is countable dense subset).

Automorphism groups

- Suppose M is a first order L-structure (with the underlying set Ω). Let $Aut(M)$ be the automorphism group of M.
- Consider $Aut(M)$ as a group acting on M (and M^n).
- It is clear that $Aut(M)$ is a subgroup of $Sym(\Omega)$.

Fact 1.1. Let G be a subgroup of $Sym(\Omega)$. Then G is closed if and only if G is the automorphism group of some first order structure.

Proof. $\Leftarrow$. Take an elements $g \in Sym(\Omega) \backslash Aut(M)$. Since it is not an automatism there is a finite A such that $g \restriction_A$ is not an isomorphism. Then $O_f \cap Aut(M) = \emptyset$.
$\rightarrow$ For each orbit on Ω^n consider a predicate which the realisation is exactly all the elements of that orbit. Let L be the set of all those predicates. Show that G is the automorphism group of that structure in L. (Exercise) □

- The structure obtained above over Ω is called the canonical structure for G on Ω.

Automorphism groups

- When M is a countable structure we have $Aut(M)$ is a Polish group i.e. separable and completely metrizable.
- Fix an enumeration on Ω and define $d(g, h) = \frac{1}{2^i}$ where i is the smallest index with $g(i) \neq h(i)$ and $g^{-1}(i) \neq h^{-1}(i)$.
- Every open subgroup is has countable index.

Fact 1.2. Suppose G is a closed subgroup of $Sym(\Omega)$. Then either identity is isolated or $|G| = 2^{\aleph_0}$.

Proof. By Cantor's argument every non-empty perfect complete space contains a Cantor set. □

Different topologies on $Sym(\Omega)$

- When M is uncountable (Ω uncountable) then a more natural topology is the topology with basis around identity G_A where $|A| < |\Omega|$.
- Is this the only group topology on $Sym(\Omega)$.
- Discrete topology.
- Markov topology: Given a group G the Markov topology τ_M, is the intersection of all Hausdorff group topologies on G (not necessarily a group topology).
- Zariski topology: Given a group G the Zariski topology τ_Z, is generated by the subbase consisting of the sets $\{x \in G \mid x^{\epsilon_1} g_1 x^{\epsilon_2} g_2 \cdots x^{\epsilon_n} g_n \neq 1\}$, where $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

Theorem 1.3 (Guran 1967, Banakh , Guran, Protasov 2012). When Ω is countable, $\tau_Z = \tau_M = \tau_{pt}$ on $Sym(\Omega)$.

$\aleph_0$-categorical structures

- If T is $\aleph_0$-categorical then all its countable models are saturated.

Theorem 1.4 (Engeler, Ryll-Nardzewski, Svenonius). Let M be a countable first order structure and $T = Th(M)$. TFAE

- *M is an $\aleph_0$-categorical structure;*
- *$Aut(M)$ has finitely many orbits on M^n for all n (called Oligomorphic);*
- *$S_n(T)$ for all $n \in \mathbb{N}$ is finite;*
- *All the types in $S_n(T)$ are isolated.*

Corollary 1.5. Automorphism group of countable $\aleph_0$-categorical has cardinality $2^{\aleph_0}$.

Proof. Exercise. Use the theorem above and Fact 1.2. □

Fraïssé construction

Definition 1.6. A countable structure in a relational language is called homogeneous if every isomorphism between finite substructures extends to an automorphism.

Fact 1.7. If M is homogeneous over a finite relational language then M is $\aleph_0$-categorical.

Definition 1.8. An *Age* is the class of finite L-structures, closed under isomorphism.

Proof. We want to show if $N \equiv M$ and N countable then it is isomorphic to M.
We use Back and Forth argument (Exercise).
Hint: Key point any finite substructure A of M , there is a sentence expressing that A embeds in M, so $Age(M) = Age(N)$. □

Definition 1.9. Suppose L is relational language (at most countable) and $\mathcal{C}$ a countable class of finite L-structures is a Fraïssé class if it is closed under *substructure* (HP) and *isomorphism* and:

1. (JEP) If $A, B \in \mathcal{C}$ then there is $C \in \mathcal{C}$ such that $A, B \subseteq C$;
2. (AP) If $A, B, C \in \mathcal{C}$ and $A \subseteq B, C$, then there is $D \in \mathcal{C}$ such that $B \subseteq D$ and $C \subseteq D$.

We assume $\emptyset \in \mathcal{C}$ (JEP is implied by AP).

Theorem 1.10. (Fraïssé 1953) Suppose $\mathcal{C}$ is a Fraïssé class. Then, there exists a unique, up to isomorphism, countable structure M that is homogeneous and $\mathcal{C} = Age\,(M)$.

- M is called the Fraïssé limit of $\mathcal{C}$, denoted by $Flim\,(\mathcal{C})$.

Fraïssé limits

Examples 1.11.

- Class of all finite linearly ordered sets; Fraïssé limit is $(\mathbb{Q}, <)$.
- Class of all finite graphs; Fraïssé limit is called the Random graph. Class of all finite K_n-free graphs; Random K_n-free graph.
- Class of all finite metric spaces with rational distances: Let $L = \{R_q : q \in \mathbb{Q}^{\geq 0}\}$ where R_q is binary. A metric space (X, d) with rational distance is an L-structure $R_q(a, b)$ if and only if $d(a, b) \leq q$ for $q \in Q^{\geq 0}$. For AP given a subspace A of $B_1 \cap B_2$, arrange $B_1 \cup B_2$ (disjoint over A) such that $d(x_1, x_2) = \min_{a \in A}(d(x_1, a) + d(x_2, a))$ for $x_i \in B_i$. Fraïssé limit is called rational Urysohn space which the completion is the Urysohn universal space (metric space that contains all separable metric spaces).

Fact 1.12. *A countable structure M with $Age\,(M) = \mathcal{C}$ is homogeneous if and only if $A \subseteq M$ and $A \subseteq B \in \mathcal{C}$, then there is an embedding $g : B \longrightarrow M$ with $g \restriction_A = 1_A$ and $gB \subseteq M$ (called richness).*

Proof. Exercise. □

Proof

The uniqueness follows by a back-and-forth argument. For existence, we aim to build a rich structure M. Essentially, there are countably many such triples (f, A, B) to consider. We build M as a union of a chain $M_0 \subseteq M_1 \subseteq M_2 \ldots$ of members of $\mathcal{C}$ When doing this we ensure that:

1. If $A \in \mathcal{C}$ then $A \subseteq M_i$ for some i;
2. If (f, A, B) is triple and $A \subseteq M_i$ there is some $j > i$ such that $B \to M_j$ and

To fulfil (1) we use JEP.
For (2): suppose the construction has reached stage $k > i$. At the next stage we can take M_{k+1} which is the the amalgamation of $A \to M_k$ (inclusion), (f, A, B) i.e. sing AP we obtain $h : M_k \to M_{k+1}$ (which can be taken as inclusion), and $g : B \to M_{k+1}$ with $gfa = ha = a$ for all $a \in A$

Lecture 2

Applications

Various questions on automorphism groups

- Small index property and reconstruction problem.
- Simple groups
- Extremely amenable groups

Small index property

Definition 2.1. We say a countable structure M it has the small index property (SIP) if whenever H is a subgroup of $Aut(M)$ of index less $< 2^{\aleph_0}$, then H is open.

- The SIP implies that we can recover the topology on G from its group-theoretic structure: the open subgroups are precisely the subgroups of small index and the cosets of these form a base for the topology.
- SIP implies group isomorphism is a homeomorphism. When ω-categorical we can recover the structure up to bi-interpretability (Ahlbrandt and Ziegler, Rubin).

Examples 2.2.

- (Dixon, Neumann, Thomas) Countable infinite set Ω.
- (Truss)$(\mathbb{Q}, <)$
- (Evans) General linear groups and classical groups over countable fields.
- (Hodges, Hodkinson, Lascar, Shelah) Random graph and $\aleph_0$-categorical ω-stable structures.

- The usual technique to show M has SIP is to show $Aut(M)$ has ample homogenous automorphisms. In order to show that one uses a combinatorial condition called EPPA.
- (Lascar, Shelah) When M is uncountable and saturated then it has SIP.

Simple groups

- (Higman 1954) The non-trivial, proper normal subgroups of $G = Aut(\mathbb{Q}; \leq)$ are,
$$L(\mathbb{Q}) = \{g \in G : \exists q \in \mathbb{Q}, g \restriction_{(-\infty,q)} = 1_G\},$$
$$R(\mathbb{Q}) = \{g \in G : \exists q \in \mathbb{Q}, g \restriction_{(a,\infty)} = 1_G\},$$
and $B(\mathbb{Q}) = L(\mathbb{Q}) \cap R(\mathbb{Q})$.
- (Truss 1985) The automorphism group of countable Random graph is simple.
- (Droste, Holland, Macpherson 1989) The automorphism group of a countable, homogeneous semilinear order has many normal subgroups ($2^{2^{\aleph_0}}$ many).
- (Lascar 1992) Let M be a countable saturated model that $M = acl(D)$ where D is a strongly minimal set. The strong automorphism group M modulo the bounded automorphism groups of M is simple.

Simple groups

Corollary 2.3 (Lascar). If F and K are algebraically closed filed of characteristic zero such that $K \subseteq F$ and the transcendental degree of F over K is strictly bigger than $\aleph_0$, then the automorphism group of F that fixes K point-wise is a simple group. Especially, $Aut_{\mathbb{Q}}(\mathbb{C})$ is a simple group.

- (Macpherson, Tent 2011) The automorphism group of a free-homogeneous structures, which is transitive and not equal to the full-symmetric group is a simple group.
- (Tent, Ziegler 2012) Suppose M is a countable structure with a *local* stationary independence relation and assume $g \in G = Aut(M)$ move maximally. If G contains a dense conjugacy class, then any element of G is the product of eight conjugates of g.
- (Tent, Ziegler 2012) The isometry group of of the Urysohn space is simple modulo the normal subgroup of bounded isometries.

Stationary independence

Consider $\mathop{\smile}\limits^{\vert}$ to be ternary relation between finite subsets $A, B, C \subseteq M$; written $A \mathop{\smile}\limits^{\vert}_B C$ and pronounced 'A is independent from C over B'.

Definition 2.4. We say $\mathop{\smile}\limits^{\vert}$ is a stationary independence if the following holds

- Invariance: If $g \in G$ and $A \mathop{\smile}\limits^{\vert}_B C$ then $gA \mathop{\smile}\limits^{\vert}_{gB} gC$.
- Monotonicity: If $A \mathop{\smile}\limits^{\vert}_B C \cup D$, then $A \mathop{\smile}\limits^{\vert}_B C$ and $A \mathop{\smile}\limits^{\vert}_{B \cup C} D$.
- Transitivity: If $A \mathop{\smile}\limits^{\vert}_B C$ and $A \mathop{\smile}\limits^{\vert}_{B \cup C} D$, then $A \mathop{\smile}\limits^{\vert}_B C \cup D$.
- Symmetry: If $A \mathop{\smile}\limits^{\vert}_B C$ then $C \mathop{\smile}\limits^{\vert}_B A$.
- Existence: There exists $g \in G_B$ such that $g(A) \mathop{\smile}\limits^{\vert}_B C$.
- Stationarity: Suppose A_1 , A_2 , B, C are finite structures with $B \subseteq A_i$, $A_1 \equiv_B A_2$ and $A_i \mathop{\smile}\limits^{\vert}_B C$ then $A_1 \equiv_{BC} A_2$.

If it is only defined for non-empty base then it is called local.

Examples 2.5.

- Pure set and Random graph.

Definition 2.6.

- Let $\mathop{\smile\!\!\!\!\!\!\mid}$ be a (local) independence relation on M and let $g \in G$. We say that g moves maximally if, for all (non-empty) finite sets A and all types p over X, there is a realisation a of p such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{A;gA} ga$ (i.e. $a \mathop{\smile\!\!\!\!\!\!\mid}_A gA \cup \{ga\}$ and $a \cup A \mathop{\smile\!\!\!\!\!\!\mid}_{gA} ga$).
- G contains a dense conjugacy class of for every finite tuples $\bar{x}, \bar{y}, \bar{a}, \bar{b}$ with $tp(\bar{x}) = tp(\bar{y})$ and $tp(\bar{a}) = tp(\bar{b})$ there are tuples $\bar{x}'\bar{y}'$ such that $tp(\bar{x}', \bar{y}') = tp(\bar{x}, \bar{y})$ and $tp(\bar{x}', \bar{a}) = tp(\bar{y}', \bar{b})$.

Theorem 2.7 (Tent, Ziegler 2012). Suppose M is a countable structure with a local stationary independence relation and assume $g \in G = Aut(M)$ move maximally. If G contains a dense conjugacy class, then any element of G is the product of eight conjugates of g.

Idea of the proof

- Consider $\phi : G^4 \to G$ be $\phi(h_1, \dots, h_4) := g^{h_1} g^{h_2} g^{h_3} g^{h_4}$.
- Show that $\phi(G^4)$ has is not meager.
- Since G contains a dense conjugacy class, using some Baire category argument we can show that $\phi(G^4)$ is not co-meager.
- g^{-1} moves maximally. Consider $\phi' : G^4 \to G$ be $\phi(h_1, \dots, h_4) := g^{-h_1} g^{-h_2} g^{-h_3} g^{-h_4}$.
- Given f we get $\phi(G^4) f \cap \phi'(G^4) \neq \emptyset$.

Definition 2.8.

- Let G be a topological group. A continuous action of G on a compact Hausdorff space is called a G-flow.
- Group G is called extremely amenable if every G-flow has a fix point.
- Group G is amenable if every G-flow supports a G-invariant Borel probability measure.

Examples 2.9.

- If G is a subgroup of $Sym(\Omega)$ then every closed G-invariant subset of $\{0,1\}^{\Omega^2}$ is a G-flow.
- $LO(\Omega) := \{R \in \{0,1\}^{\Omega^2} : R \text{ is a linear order on } \Omega\}$.

Fact 2.10. If G is an extremely amenable then there is a G-invariant linear order on Ω.

Ramsey property

Theorem 2.11. (Kechris, Pestov, Todorcevic 2005) Let $G \leq Sym(\Omega)$ be a closed subgroup. Then the following are equivalent:

1. *G is extremely amenable.*
2. *$G = Aut\,(M)$, where M is the Fraïssé limit of a ordered Fraïssé class with the Ramsey property.*

Examples 2.12.

- Class of all finite linearly ordered sets.
- (Nešetřil and Rödl) Class of all finite ordered graphs.
- (Nešetřil) Class of all finite ordered metric spaces with rational distances.
- (Dual Ramsey theorem of Graham-Rothschild) The class of all finite Boolean algebras $(B; \vee, \wedge, -^c, 0, 1)$ has the Ramsey property.

Ramsey property

- Suppose $\mathcal{C}$ is a class of L-structures.
- Let $A, B \in \mathcal{C}$ and $A \subseteq B$. Let $\binom{B}{A}$ be the set of all embeddings of $A \text{ in } B$.
- Write $C \to (B)^A_k$ when for every $f : \binom{C}{A} \to \{0, 1, \ldots, k-1\}$ there is $B' \in \binom{C}{B}$ such that $\left\{ f(A') : A' \in \binom{B'}{A} \right\}$ is monochromatic.

Definition 2.13. Suppose $\mathcal{C}$ is a class of L-structures. $\mathcal{C}$ is a Ramsey class if for every $A, B \in \mathcal{C}$ and $k \in \mathbb{N}$ with $A \subseteq B$ there is $C \in \mathcal{C}$ such that $C \to (B)^A_k$.

Theorem 2.14 (Classical infinite Ramsey Theorem). *Let B be a countably infinite set, and let m, r be finite integers. For every* $h : \binom{B}{m} \to \{0, \ldots, r-1\}$ *there exists an infinite* $P \subseteq B$ *such that is monochromatic on all m-element subsets of P.*

Corollary 2.15 (Finite Ramsey Theorem). *For every* $k, m, n \in \mathbb{N}$ *there is l such that* $l \to (k)^n_m$.

Proof. Suppose not. Then we have bad coloring for an n, m. Find a bad coloring on an infinite set using König's Lemma (infinite finite branching tree an infinite path). □

Proposition 2.16. *Let* $\mathcal{C}$ *be an amalgamation class, and let* $M = Flim(\mathcal{C})$. *Then* $\mathcal{C}$ *has the Ramsey property if and only if* $M \to (B)^A_k$ *for all* $k \geq 2$ *and for all* $A, B \in \mathcal{C}$.

Main question

Question

Let M be a homogeneous structure with over a finite relational language. Then there is a homogeneous expansion M^* of M with finite relational language whose age has the Ramsey property.

Theorem 2.17 (Evans, Hubička, Nešetřil). *There exists a closed oligomorphic permutation group without a closed oligomorphic extremely amenable subgroup. Equivalently, there exists an $\aleph_0$-categorical structure without an ordered $\aleph_0$-categorical Ramsey expansion.*

Theorem 2.18. Let $\mathcal{C}$ be a class of ordered finite L-structures that is closed under substructures, isomorphism, and has JEP. If $\mathcal{C}$ is Ramsey, then it has AP.

Proof. Let A, B_1, B_2 be elements of $\mathcal{C}$ such that $A \subseteq B_i$. Since $\mathcal{C}$ has JEP, there exists a structure $C \in \mathcal{C}$ with embeddings $e_i : B_i \to C$. If e_1, e_2 have the same restriction to A, we are done, so assume otherwise.
Let $D \in \mathcal{C}$ be such that $D \to (C)^A_2$. Define a coloring $h : \binom{C}{A} \to \{0, 1\}$ as follows: for $A' \in \binom{C}{A}$ define $h(A') = 1$ if there exists $C' \in \binom{D}{C}$ and let $gC = C'$ such that $ge_1A = A'$; otherwise define $h(A') = 0$. By the Ramsey property, there is $C_0 \in \binom{D}{C}$ that is monochromatic. Let $tC = C_0$. Note that according to the coloring, h is constant 1 for elements $\binom{C_0}{A}$. Consider te_2A then $h(te_2A) = 1$. We are done.

□

Idea of the proof

Assume G is extremely amenable then show:
(2) For every open subgroup H of G and if $c : G/H \to \{0, \ldots, k-1\}$ and $A \subseteq G/H$ is finite, there is $g \in G$ and $i < k$ such that $c(ga) = i$ for all $a \in A$.
Then we show (2) is equivalent to (3)
(3) G preserves a linear ordering on Ω and G has the Ramsey property. Consider the G-flow $\{0, \cdots, k-1\}^{G/H}$. Let Y be the closure in this of the G-orbit $\{gc : g \in G\}$. This is a G-flow, so must contain a G-fixed point. So it contains a constant function $f_i(z) = i$ (for some $i < k$). In other words, f_i is in the closure of $\{gc : g \in G\}$. This translates into the condition in (2).

Suppose not and there are $A \subseteq B$ such that $C \nrightarrow (B)^A_2$ for all $C \in \mathcal{C}$. Pick $C_0 \in \binom{\mathrm{M}}{C}$. Then, for every finite $C_0 \subseteq E \subseteq \mathrm{M}$ there exists a 2-coloring $c_E : \binom{E}{A} \longrightarrow \{0,1\}$ that is not constant on any copy of B. Take $\mathcal{I}$ be the set of all finite subsets of M, as an index set, and for $D \in \mathcal{I}$, let $\mathcal{X}_D := \{F \in \mathcal{I} : D \subseteq F\}$.One can show $\mathcal{E} := \{\mathcal{X}_A : A \in \mathcal{I}\}$ has FIP. Hence, there exists an ultra-filter $\mathcal{U}$ on the index set $\mathcal{I}$ such that for every finite $D \in \mathcal{I}$ the set $\mathcal{X}_D \in \mathcal{U}$. Define a 2-coloring $c : \binom{\mathrm{M}}{A} \longrightarrow \{0,1\}$ as follows: for $A' \in \binom{\mathrm{M}}{A}$

$$c(A') := i \Leftrightarrow \{E \in \mathcal{I} : A'B_0 \subseteq E \text{ and } c_E(A') = i\} \in \mathcal{U}.$$

Using (2) show that there is a $B' \in \binom{M}{B}$ that is monochromatic; contradiction