Introduction to o-minimality and applications
An excursion into Model Theory and its applications
LMS online lecture series - Fall 2020

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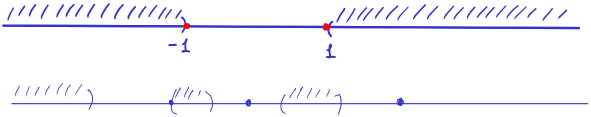
Part I: o-minimality

Ordered field of the reals

- Consider the ordered field of the reals $(\mathbb{R}; +, \cdot, <, 0, 1)$.
- The formula $\varphi(x) := \exists y (x^2 - 1 \geq y^2)$ defines the set $(-\infty, -1) \cup \{-1\} \cup \{1\} \cup (1, \infty)$.
- By quantifier elimination any formula $\varphi(x)$ is equivalent to a Boolean combination of formulas of the form $p(x) = 0$ and $p(x) > 0$ where $p(X) \in \mathbb{R}[X]$. Hence every definable set in $\mathbb{R}$ is a finite union of points and open intervals.
- This means that all definable sets in one variable can be defined (with parameters) in the language $\{<\}$.
- Structures with this property are said to be *o-minimal*.



Conventions

- Throughout, $\mathcal{M} := (M; <, \ldots)$ will be a structure with $(M; <) \models \mathrm{DLO}$.
- An *interval* is an open interval with endpoints in $M \cup \{\pm\infty\}$.
- *Definable* means definable with parameters.
- For a function f its graph is denoted by $\Gamma(f)$.
- Let $X \subseteq M^n$. A function $f : X \to M^k$ is definable if $\Gamma(f)$ is a definable subset of M^{n+k}.
- There is a natural topology on M – the order topology. On M^n we use the product topology.

Definition of o-minimality

Definition

$\mathcal{M} = (M; <, \dots)$ is *o-minimal* if every definable subset of M is a finite union of points and intervals.

Example

- $(\mathbb{Q}; <)$, $(\mathbb{R}; <)$
- $(\mathbb{Q}; <, +)$
- $(\mathbb{R}; +, \cdot, <)$

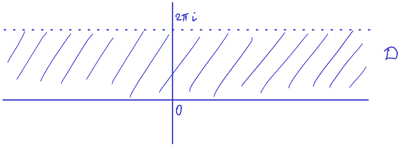
Example (Non-examples)

- $(\mathbb{R}; +, \cdot, \sin, <)$ 
- $(\mathbb{Q}; +, \cdot, <)$
- $\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$ (here we identify $\mathbb{C}$ with $\mathbb{R}^2$)

The topology on an o-minimal structure is "tame".

Further examples

- Let $\mathbb{R}_{\text{an}}$ be the expansion of $(\mathbb{R}; +, \cdot, <)$ by restricted analytic functions: for each real analytic function defined on an open set containing $[0,1]^n$ we have a function symbol for its restriction to $[0,1]^n$. This is o-minimal.
- $\sin|_{[0,2\pi]}$ is definable in $\mathbb{R}_{\text{an}}$, for $\sin(2\pi x)|_{[0,1]}$ is definable.
- More generally, if $f : U \to \mathbb{R}$ is an analytic function defined on an open domain $U \subseteq \mathbb{R}^n$ and $B \subseteq U$ is a bounded closed box then $f|_B$ is definable in $\mathbb{R}_{\text{an}}$.
- Is $\sin\left(\frac{1}{x}\right)|_{(0,1)}$ definable in $\mathbb{R}_{\text{an}}$?
- $\mathbb{R}_{\text{exp}} := (\mathbb{R}; +, \cdot, \exp, <)$ is o-minimal (Wilkie, 1996).
- $\mathbb{R}_{\text{an,exp}}$ is the expansion of $\mathbb{R}_{\text{an}}$ by the exponential function $\exp : \mathbb{R} \to \mathbb{R}^{>0}$. This is also o-minimal.
- Let $D := \{z \in \mathbb{C} : 0 \leq \operatorname{Im} z < 2\pi\}$. Then the restriction of the complex exponentiation to D is definable in $\mathbb{R}_{\text{an,exp}}$.



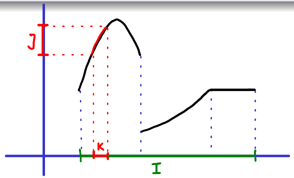
Theorem (Monotonicity theorem)

Let $f : I \to M$ be a definable function on an interval $I = (a, b)$. Then there are points $a = a_0 < a_1 < \ldots < a_n = b$ such that on each interval (a_i, a_{i+1}) the function f is either constant or strictly monotonic and continuous.

Sketch proof.

It suffices to show that for any definable function $f : I \to M$ there is a subinterval of I on which f is constant or strictly monotonic and continuous. Indeed, let $X \subseteq I$ be the set of all points x such that f is constant or strictly monotonic and continuous on a neighbourhood of x. If $I \setminus X$ is infinite then it contains an interval which is a contradiction. So $I \setminus X$ is finite and we are done.

We prove that on an infinite subinterval f is constant or injective. We may assume all fibres $f^{-1}(y)$ are finite, for otherwise f would be constant on a subinterval. Then $f(I)$ is infinite and so contains an interval J. Define $g : J \to I$ by $g(y) := \min f^{-1}(y)$. Then g is injective and the image $g(J)$ contains an interval K. Hence, $f|_K$ is injective. □



Uniform finiteness

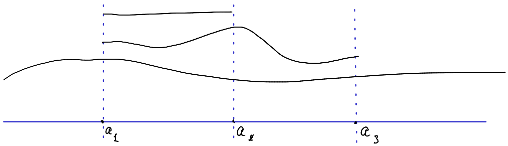
For $Y \subseteq M^{n+1}$ and $\bar{a} \in M^n$ let $Y_{\bar{a}} := \{y \in M : (\bar{a}, y) \in Y\}$.

Theorem

Let $Y \subseteq M^2$ be a definable set. Then there is a number N such that for any $a \in M$ if Y_a is finite then $|Y_a| \leq N$.

Exercise

Let $Y \subseteq M^2$ be definable such that Y_a is finite for each a. Show that there are points $-\infty = a_0 < a_1 < \ldots < a_{k+1} = +\infty$ such that the intersection of Y with each vertical strip $(a_i, a_{i+1}) \times M$ has the form $\Gamma(f_{i,1}) \cup \ldots \cup \Gamma(f_{i,m_i})$ where each $f_{i,j} : (a_i, a_{i+1}) \to M$ is a definable continuous function and with $f_{i,1}(x) < \ldots < f_{i,m_i}(x)$ for all $x \in (a_i, a_{i+1})$.



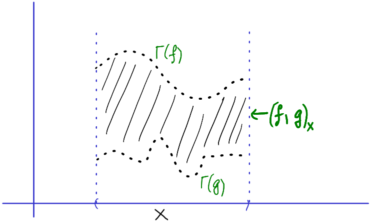
- For a definable set $X \subseteq M^n$ let $C(X) := \{f : X \to M : f \text{ is definable and continuous}\}$. Let also $C_\infty(X) = C(X) \cup \{-\infty, +\infty\}$ where $-\infty, +\infty$ are regarded as constant functions on X.
- For $f, g \in C_\infty(X)$ write $f < g$ if $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in X$. In this case define $(f, g)_X := \{(\bar{x}, y) \in X \times M : f(\bar{x}) < y < g(\bar{x})\}$.

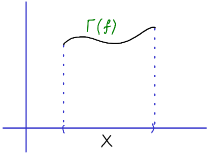
Definition

Let $\bar{i} := (i_1, \ldots, i_m) \in \{0, 1\}^m$. An $\bar{i}$-cell is a definable subset of M^m defined inductively on m as follows.

- A (0)-cell is a point and a (1)-cell is an open interval in M.
- Suppose $\bar{i}$-cells have been defined. Then an $(\bar{i}, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$ where X is an $\bar{i}$-cell. An $(\bar{i}, 1)$-cell is a set of the form $(f, g)_X$ where X is an $\bar{i}$-cell and $f, g \in C_\infty(X)$ and $f < g$.

A *cell* is an $\bar{i}$-cell for some $\bar{i}$.





Definition

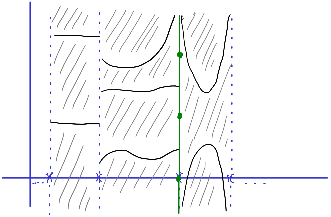
A *decomposition* of M^n is a partition of M^n into finitely many cells defined as follows by induction.

- A decomposition of M is a partition of M into a union of finitely many disjoint cells.
- A decomposition of M^{n+1} is a partition of M^{n+1} into finitely many cells the projections of which to the first n coordinates form a decomposition of M^n.

Theorem

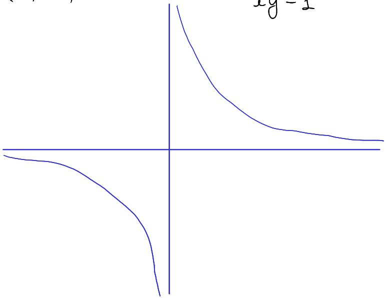
I_n *For any definable sets $A_1, \ldots, A_k \subseteq M^n$ there is a decomposition of M^n which partitions each A_i.*

II_n *Given a definable function $f : X \to M$ with $X \subseteq M^n$, there is a decomposition of M^n partitioning X such that for any cell $C \subseteq X$ the restriction $f|_C : C \to M$ is continuous.*

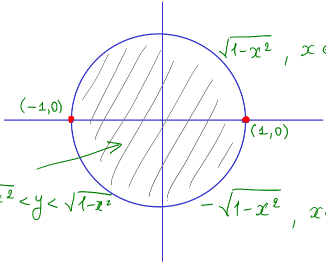


Definable sets in $\mathbb{R}^2$

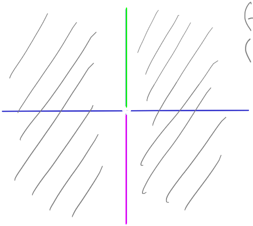
① $(\mathbb{R}; +, \cdot, <)$ $\quad xy = 1$



② $(\mathbb{R}; +, \cdot, <)$ $\quad x^2 + y^2 \leq 1$



③ $(\mathbb{R}; +, \cdot, <)$ $\quad \mathbb{R}^2 \setminus \{0\}$



$(-\infty, 0) \times \mathbb{R}$

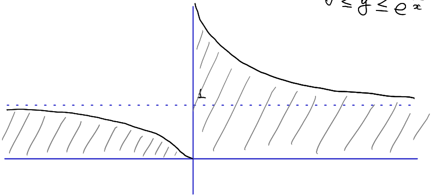
$(0, +\infty) \times \mathbb{R}$

$\{0\} \times (0, +\infty)$

$\{0\} \times (-\infty, 0)$

④ $(\mathbb{R}; +, \cdot, \exp, <)$ $\quad \exists z (xz = 1 \wedge 0 \leq y \leq e^z)$

$0 \leq y \leq e^{\frac{1}{x}}$



Theorem

Let $Y \subseteq M^{n+1}$ be a definable set. Then there is a number k such that for any $\bar{a} \in M^n$ if $Y_{\bar{a}}$ is finite then $|Y_{\bar{a}}| \leq k$. Hence, the quantifier $\exists^{\infty}$ is first-order expressible.

Theorem

Let $\mathcal{M}$ and $\mathcal{N}$ be elementarily equivalent ordered structures. If $\mathcal{M}$ is o-minimal then so is $\mathcal{N}$.

Proof.

Let $\phi(x, \bar{b})$ define a set $X_{\bar{b}}$ in N. The boundary of $X_{\bar{b}}$ is definable (uniformly in $\bar{b}$) by a formula $\psi(x, \bar{b})$. For every $\bar{a} \in M^{|\bar{b}|}$ the formula $\psi(x, \bar{a})$ defines the boundary of $\phi(x, \bar{a})$ and is finite. By uniform finiteness, $\psi(x, \bar{a})$ has at most k elements for some k independent of $\bar{a}$. This is part of the theory of $\mathcal{M}$, hence also of the theory of $\mathcal{N}$. Thus, $\psi(x, \bar{b})$ has at most k elements, which means $X_{\bar{b}}$ is a union of finitely many points and intervals. □

$\varphi(x, \bar{a})$

Definition

- A subset $X \subseteq M^n$ is *definably connected* if there are no definable open sets U_1, U_2 such that $X \subseteq U_1 \cup U_2$, $X \cap U_1 \cap U_2 = \emptyset$ and $X \cap U_1 \neq \emptyset, X \cap U_2 \neq \emptyset$.
- For a definable set $X \subseteq M^n$ a *definably connected component* of X is a maximal definably connected subset of X.

Proposition

Every definable set $X \subseteq M^n$ has finitely many definably connected components. They are definable, open and closed in X and form a partition of X.

Proof.

Let $X = \cup_i C_i$ be a cell decomposition of X, and let Y be a definably connected component of X. Each C_i is definably connected, hence either $C_i \subseteq Y$ or $C_i \cap Y = \emptyset$. Therefore, Y is a union of cells. $\square$

Proposition

In a parametric family of definable sets the number of connected components is bounded.

Dimension

Definition

For a definable set X let $\dim X := \max\{i_1 + \ldots + i_m : X \text{ contains an } (i_1, \ldots, i_m)\text{-cell}\}$. We also set $\dim \emptyset = -\infty$.

- A definable set has dimension 0 if and only if it is finite.
- $\dim M^n = n$.
- Let $X \subseteq M^n$ be definable. Then $\dim X$ is the largest integer k for which some projection of X to M^k has non-empty interior in M^k.

Definition

For a subset $A \subseteq M$ the *algebraic closure* of A is the union of all finite definable sets over A, and the *definable closure* of A is the union of all definable singletons over A. For instance, in $(\mathbb{C}; +, \cdot)$ we have $\sqrt{2} \in \operatorname{acl}(\mathbb{Q}) \setminus \operatorname{dcl}(\mathbb{Q})$, while in $(\mathbb{R}; +, \cdot)$ we have $\sqrt{2} \in \operatorname{dcl}(\mathbb{Q})$.

Theorem

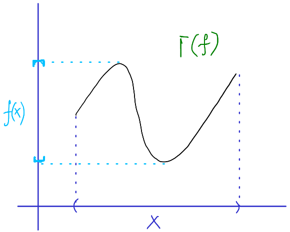
In an o-minimal structure $\operatorname{acl} = \operatorname{dcl}$, *and this operator defines a pregeometry. Its dimension agrees with the dimension function defined above.*

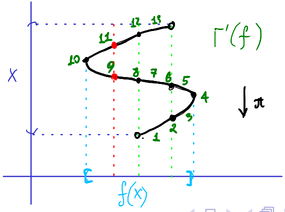
Theorem

Let $X \subseteq M^n$ be definable and let $f : X \to M^k$ be a definable map such that for any $x \in X$ the fibre $f^{-1}(f(x))$ is finite. Then $\dim f(X) = \dim X$.

Sketch proof.

Let $\Gamma'(f) := \{(f(x), x) : x \in X\}$ and let $\pi : \Gamma'(f) \to f(X)$ be the projection map. Observe that the map $x \mapsto (f(x), x)$ is a definable bijection from X to $\Gamma'(f)$, hence $\dim X = \dim \Gamma'(f)$. Write $\Gamma'(f) = \cup_i C_i$ using cell decomposition. For each cell C_i the projection $\pi(C_i)$ is a cell and for $y \in \pi(C_i)$ the fibre $\{x \in X : (y, x) \in C_i\}$ is also a cell. Since it is finite, it must be a singleton. Therefore, π is a bijection from C_i to $\pi(C_i)$, so $\dim C_i = \dim \pi(C_i)$. Hence $\dim f(X) = \dim \Gamma'(f)$. □





Let $\mathcal{M} = (M; <, \ldots)$ be an o-minimal structure.

1. Find a cell decomposition of $\mathbb{R}^2 \setminus X$ where X is a finite set.
2. Does the cell decomposition theorem hold for infinitely many definable sets $A_1, A_2, \ldots$?
3. Let $\pi : M^{n+k} \to M^k$ be the projection on the firs n coordinates. Prove that if $C \subseteq M^{n+k}$ is a cell and $a \in \pi C$ then $C_a = \{y \in M^k : (a, y) \in C\}$ is a cell.
4. Show that a cell in M^n of dimension n is open.
5. Show that cells are definably connected.
6. Show that if $\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}; <)$ then a definable set $X \subseteq \mathbb{R}^k$ is connected if and only if it is definably connected.
7. Let $X \subseteq M^n$ be definable. Show that $\dim(\bar{X} \setminus X) < \dim X$, where $\bar{X}$ is the topological closure of X.
8. Show that if $X \subseteq M^n$ is a cell of dimension k then it is definably homeomorphic to an open subset of M^k.
9. Show that if $X \subseteq M^n$, $Y \subseteq M^k$ are definable sets and there is a definable bijection between them then $\dim X = \dim Y$.
10. Let $X, Y \subseteq M^n$ be definable. Show that $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$.

Part II: Applications

Holomorphic maps with discrete fibres

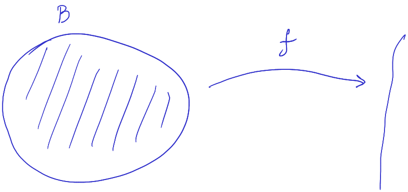
Theorem

Let $U \subseteq \mathbb{C}^n$ be an open domain and let $f : U \to \mathbb{C}^n$ be a holomorphic map all fibres of which are discrete. Then $f(U)$ has a non-empty interior.

This is a weak version of Remmert's open mapping theorem.

Sketch proof.

Identify $\mathbb{C}$ with $\mathbb{R}^2$. For some box $B \subseteq U$ the restriction $f|_B$ is definable in $\mathbb{R}_{\text{an}}$. Hence, by the "fibre dimension theorem" for o-minimal structures, $\dim_{\mathbb{R}} f(B) = \dim_{\mathbb{R}} B = 2n$. Hence $f(B) \subseteq \mathbb{R}^{2n}$ contains a cell of dimension $2n$, which is open. □



Schanuel's conjecture

Conjecture (Schanuel's conjecture)

Let $z_1, \ldots, z_n \in \mathbb{C}$ be $\mathbb{Q}$-linearly independent. Then

$$\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

- Here td stands for *transcendence degree*. Recall that for two fields $K \subseteq L$, some elements $a_1, \ldots, a_n \in L$ are called algebraically independent over K if $p(a_1, \ldots, a_n) \neq 0$ for any non-zero polynomial p with coefficients from K, and $\operatorname{td}_K L$ (often denoted by $\operatorname{td}(L/K)$) is the cardinality of a maximal set of algebraically independent elements from L over K.
- Schanuel's conjecture is considered out of reach.
- Zilber explored the model theory of $\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$, and constructed algebraically closed fields of characteristic 0 with a unary function, called *pseudo-exponentiation*, which mimics some of the basic properties of the complex exponential function and satisfies an analogue of Schanuel's conjecture.
- Zilber's work gave rise to two major conjectures: the Exponential Algebraic Closedness conjecture, and the Conjecture on Intersections with Tori.
- A functional analogue of Schanuel's conjecture, known as the Ax-Schanuel theorem, can be proven using o-minimality.

Conjecture ($SC_{\mathbb{R}}$)

Let $x_1, \ldots, x_n \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Then $\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$.

- Let $T_{\mathrm{exp}} := \mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$. Tarski asked if T_{exp} is decidable. Macintyre and Wilkie proved that if Schanuel's conjecture holds for the reals then T_{exp} is decidable.
- A natural question is whether $SC_{\mathbb{R}}$ is part of T_{exp}. For this, one needs a uniform version of the conjecture.

Conjecture ($SC_{\mathbb{R}}$)

Let $V \subseteq \mathbb{R}^{2n}$ be an algebraic variety over $\mathbb{Q}$ with $\dim V < n$. *If $(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \in V$ then there are integers $m_1, \ldots, m_n$, not all zero, such that $\sum_k m_k x_k = 0$.*

Conjecture (Uniform $SC_{\mathbb{R}}$)

Let $V \subseteq \mathbb{R}^{2n}$ be an algebraic variety over $\mathbb{Q}$ with $\dim V < n$. *Then there is a natural number N such that if $(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \in V$ then there are integers $m_1, \ldots, m_n \in [-N, N]$, not all zero, such that $\sum_k m_k x_k = 0$.*

Theorem (Kirby–Zilber, 2004)

Schanuel's conjecture over $\mathbb{R}$ implies its uniform version.

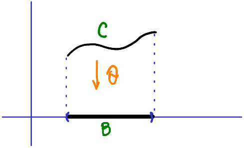
If we work in an expansion of $\mathbb{R}$ and in the definition of cells we require the functions f, g to be analytic then we get *analytic cells*. It is know that $\mathbb{R}_{\exp}$ has analytic cell decomposition.

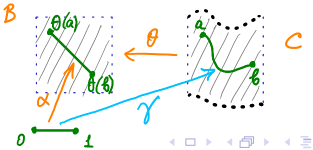
Lemma

Let $\mathcal{R}$ be an expansion of $\mathbb{R}$. If $C \subseteq \mathbb{R}^n$ is a cell of dimension m then there are an open box $B \subseteq \mathbb{R}^m$ (a product of m open intervals in $\mathbb{R}$) and a definable homeomorphism $\theta : C \to B$. If C is an analytic cell then θ can be chosen to be an analytic diffeomorphism.

Lemma

Let $\mathcal{R}$ be an expansion of $\mathbb{R}$ and let $C \subseteq \mathbb{R}^n$ be an analytic cell. For any points $a, b \in C$ there is a definable analytic path from a to b contained in C, that is, an analytic map $\gamma : [0,1] \to C$ such that $\gamma(0) = a, \gamma(1) = b$.

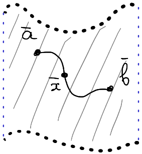




$\gamma = \theta^{-1} \circ \alpha$

Proof of the theorem

- Assume Schanuel's conjecture over $\mathbb{R}$.
- Let $V \subseteq \mathbb{R}^{2n}$ be an algebraic variety over $\mathbb{Q}$ of dimension $< n$. The set $W := \{\bar{x} \in \mathbb{R}^n : (\bar{x}, e^{\bar{x}}) \in V\}$ is definable in $\mathbb{R}_{\exp}$, hence can be decomposed into a finite union of analytic cells.
- Pick a cell $C \subseteq W$ and points $\bar{a}, \bar{b} \in C$. Let $\gamma : [0,1] \to C$ be a definable analytic path from $\bar{a}$ to $\bar{b}$ in C.
- By $\mathrm{SC}_{\mathbb{R}}$ every point $\bar{x} \in \mathrm{Im}(\gamma)$ satisfies a linear equation $\sum_k m_k x_k = 0$. Since there are countably many possible linear equations, one of them must be satisfied by infinitely many points. Thus for some linear map $h(\bar{x}) = \sum_k m_k x_k$ the set $\{t \in [0,1] : h(\gamma(t)) = 0\}$ is infinite.
- It is a definable subset of $[0,1]$, hence it must contain an interval. This means $h \circ \gamma : [0,1] \to \mathbb{R}$ is zero on an open interval. Since it is analytic, it must be identically zero on $[0,1]$. Therefore $h(\bar{a}) = h(\bar{b}) = 0$.
- We conclude that $h(\bar{x}) = 0$ for any $\bar{x} \in C$, for $\bar{a}, \bar{b}$ were arbitrary points in C.
- Since W has finitely many cells, every point of W must satisfy one of finitely many linear equations over $\mathbb{Z}$.



Theorem (Dimension of intersection)

Let $V, W \subseteq \mathbb{C}^n$ be irreducible varieties. Then any non-empty irreducible component X of the intersection $V \cap W$ satisfies $\dim X \geq \dim V + \dim W - n$.

Definition (Atypical intersection)

Let V, W be varieties in $\mathbb{C}^n$. A non-empty irreducible component X of $V \cap W$ is said to be *typical* if $\dim X = \dim V + \dim W - n$ and *atypical* if $\dim X > \dim V + \dim W - n$.

Two curves in $\mathbb{C}^2$ are likely to intersect, while two curves in $\mathbb{C}^3$ are not. When they do, we have an atypical intersection.

Definition

An *algebraic torus* is an irreducible algebraic subgroup of $(\mathbb{C}^{\times})^n$ for some positive integer n, where $\mathbb{C}^{\times}$ is the multiplicative group of C.

A variety defined by equations of the form $y_1^{m_1} \cdots y_n^{m_n} = 1$, where $m_i \in \mathbb{Z}$, is a subgroup of $(\mathbb{C}^{\times})^n$ and can be decomposed into a disjoint union of an algebraic torus (the connected component of the identity element) and its torsion cosets. For example, $y_1^3 y_2^6 = 1$ is the union of three irreducible varieties given by $y_1 y_2^2 = \zeta$ where $\zeta^3 = 1$.
Note that an algebraic torus is the image of a $\mathbb{Q}$-linear subspace of $\mathbb{C}^n$ under the exponential function.

Definition

Let $V \subseteq (\mathbb{C}^{\times})^n$ be an algebraic variety. A subvariety $X \subseteq V$ is *atypical* if it is an atypical component of an intersection $V \cap T$ where $T \subseteq (\mathbb{C}^{\times})^n$ is a torsion coset of a torus.

Conjecture (CIT)

Every algebraic variety $V \subseteq (\mathbb{C}^{\times})^n$ contains only finitely many maximal atypical subvarieties.

Some remarks

- CIT is the difference between Schanuel's conjecture (over $\mathbb{C}$) and its uniform version.
- It was posed by Zilber, then independently by Bombieri–Masser–Zannier.
- Later, Pink proposed a more general conjecture. The general form is now known as the Zilber–Pink conjecture.
- Many special cases are known, e.g. the Mordell–Lang and the Manin–Mumford conjectures.
- Many weak versions and special cases of the Zilber–Pink conjecture have been proven using o-minimality. An important ingredient of those proofs is the Pila–Wilkie counting theorem.

Definition (Height)

For $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ define $H(a/b) = \max(|a|, |b|)$, and for $\bar{x} \in \mathbb{Q}^n$ set $H(\bar{x}) = \max_i H(x_i)$.
For a set $Z \subseteq \mathbb{R}^n$ and $T > 0$ let $Z(\mathbb{Q}, T) := \{x \in Z \cap \mathbb{Q}^n : H(\bar{x}) \leq T\}$ and $N(Z, T) := |Z(\mathbb{Q}, T)|$.

Definition

For a set $Z \subseteq \mathbb{R}^n$ the *algebraic part* of Z, denoted Z^{alg}, is the union of all positive dimensional connected semi-algebraic subsets of Z.

Theorem

Let $Z \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of $\mathbb{R}$, and let $\epsilon > 0$. Then there is a constant $c = c(Z, \epsilon)$ such that for all T we have $N(Z \setminus Z^{\text{alg}}, T) \leq cT^{\epsilon}$.

Example

Let $Z \subseteq \mathbb{R}^2$ be given by $y = 2^x$. Then $Z^{\text{alg}} = \emptyset$ (why?). If $(x, y) \in Z \cap \mathbb{Q}^2$ then $(x, y) \in \mathbb{Z}^2$. Hence $N(Z, T)$ grows logarithmically in T.

An example

Theorem

If a variety $V \subseteq (\mathbb{C}^\times)^n$ contains no cosets of positive dimensional algebraic tori, then V contains finitely many torsion points, i.e. points all coordinates of which are roots of unity.

- Let $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ be the map $(z_1, \ldots, z_n) \mapsto (e^{2\pi i z_1}, \ldots, e^{2\pi i z_n})$.
- $\pi(\bar{z})$ is a torsion point in $(\mathbb{C}^\times)^n$ iff $\bar{z} \in \mathbb{Q}^n$.
- If $W \subseteq (\mathbb{C}^\times)^2$ is given by $w_1^2 w_2^3 = 1$ then $\pi^{-1}(W)$ is the union of all lines $2z_1 + 3z_2 = k,\ k \in \mathbb{Z}$. So $\pi^{-1}(W)^{\mathrm{alg}} = \pi^{-1}(W)$.
- More generally, for an algebraic variety $W \subseteq (\mathbb{C}^\times)^n$ the set $\pi^{-1}(W)^{\mathrm{alg}}$ is the union of translates of positive dimensional $\mathbb{Q}$-linear spaces contained in $\pi^{-1}(W)$.
- π is not definable in any o-minimal structure but its restriction to $F = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z < 1\}^n$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.
- Let $Z := \pi^{-1}(V) \cap F$. Then Z^{alg} is a union of intersections of translates of $\mathbb{Q}$-linear spaces with F. These are indeed semi-algebraic.
- In particular, if V does not contain any cosets of algebraic subtori then $\pi^{-1}(V)^{\mathrm{alg}} = \emptyset$ and $Z^{\mathrm{alg}} = \emptyset$.
- So the Pila–Wilkie theorem gives a bound on the number of rational points in Z of bounded height, that is, Z contains "few" rational points.
- One can get from this to a finiteness statement.