# An excursion into Model Theory and its applications; Lecture 6

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LMS online lecture series – Fall 2020

# Review

- $L$ a countable language. Assume $T$ is a complete theory.
- A type is a complete and consistent set of $L$-formulas ($n$-type in $n$-variables).
- The notion of types extends naturally to types over sets (e.g. over $A$ where $A \subseteq M$ and $M \models T$).
- A structure $M$ is $\kappa$-saturated if for every $A \subseteq M$ of size $|A| < \kappa$, every type over $A$ is realised in $M$.
- Given $M$ and $\kappa$ infinite, there is a $\kappa$-saturated $M' \succ M$.
- A type $p(\bar{x})$ is isolated if it is isolated by a formula $\psi(\bar{x})$ if $T \cup \{\psi(\bar{x})\}$ is consistent and
$$T \vdash \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x})).$$
- By Omitting Type Theorem if $p(\bar{x})$ is not isolated there is countable model of $T$ that omits $p(\bar{x})$.

# $\aleph_0$-categoricity

Assume $L$ is countable language.

*Proposition 6.1. A countable complete theory with infinite models is $\aleph_0$-categorical if and only if all its countable models are $\aleph_0$-saturated.*

Proof. $\Longrightarrow$ Suppose $T$ has a countable model $M$ that is not saturated. That means there is a non-isolated type over a finite set in $M$. Find an elementary extension of $M$ of countable size that realises the type. This contradicts $\aleph_0$-categoricity.
$\Longleftarrow$ Back and forth. □

*Corollary 6.2. $T$ is $\aleph_0$-categorical if and only if for every all types are isolated.*

Exercise 6.3. If $M$ is an $L$-structure and $\bar{a} \in M^n$. Then $Th(M)$ is $\aleph_0$-categorical if and only if $Th(M, \bar{a})$ is $\aleph_0$-categorical.

*Corollary 6.4. If $T$ is $\aleph_0$-categorical for every all types over finite sets are isolated.*

# Space of types

Definition 6.5. Let $T$ be a theory and $n \in \mathbb{N}$. Denote $S_n(T)$ for the set of all $n$-types of $T$. When $A \subseteq M \models T$, $M$ infinite let $S_n(M; A)$ denote the set of all $n$-types over $A$.

*Proposition 6.6. The set $S_n(T)$ endow a topology: For any L-formula $\phi(\bar{x})$ let*

$$[\phi] := \{p \in S_n(T) : \phi(\bar{x}) \in p\}.$$

*The collection of $[\phi]$ form a basis for a topology on $S_n(T)$.*

- $[\bot] = \emptyset$ *and* $[\top] = S_n(T)$;
- $[\phi] \cap [\psi] = [\phi \wedge \psi]$;
- $[\phi] \cup [\psi] = [\phi \vee \psi]$.

- Note that $S_n(T) \setminus [\phi] = [\neg\phi]$ i.e. basic open sets are closed (called *clopen*).
- Formula $\psi$ isolates $p$ if and only if $[\psi] = \{p\}$.
- $S_0(T)$ is the set of all consistent completions of $T$.The space $S_0(T)$ is homeomorphic to the stone space on $B_0(T)$ where $B_0(T)$ is the equivalence class $\equiv_T$ of $L$-sentences ($\phi \equiv_T \psi$ iff $T \vdash \phi \leftrightarrow \psi$).

# Space of Types

*Proposition 6.7. $S_n(T)$ is a compact, totally disconnected Hausdorff space.*

Proof. Totally disconnected follows from the fact above.
If $p_1, p_2$ are two different types then there is $\phi$ such that $p_1 \vdash \phi$ but $p_2 \vdash \neg\phi$. Then $[\phi]$ and $[\neg\phi]$ are open sets which separate $p_1$ and $p_2$.
Suppose $\{[\phi_i] : i \in I\}$ is a family of closed subsets with FIP. That means $\phi_{i_1} \wedge \ldots \wedge \phi_{i_n}$ is consistent for all $n$ hence $\{\phi_i(\bar{x}) : i \in I\}$ is consistent. Take any complete extension $p$ of $\{\phi_i : i \in I\}$. Then $\bigcap[\phi_i] \neq \emptyset$. □

*Corollary 6.8. Assume $T$ is a complete theory with infinite models in a countable language $L$. Then $T$ is $\aleph_0$-categorical if and only if $S_n(M; A)$ is finite for all finite $A$ and all $n \in \mathbb{N}$.*

Proof. By the previous argument every type in $S_n(M; A)$ is isolated. For every type $p \in S_n(M; A)$ take a formula $\phi_p$ that isolates it. Then $S_n(M; A) = \bigcup_p [\phi_p]$ (this is an open covering). By compactness of $S_n(M; A)$ we have a finite covering. Hence there are $\phi_{p_1}, \ldots, \phi_{p_n}$ such that $\bigcup_i [\phi_{p_i}] = S_n(M; A)$. □

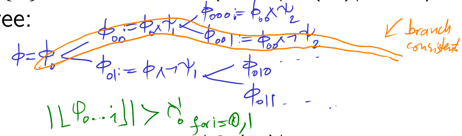
Exercise 6.9. (Ryll-Nardzewski Theorem) Assume $T$ is countable complete theory with infinite models. $T$ is $\aleph_0$-categorical if and only if there are only finitely many formulas $\phi(\bar{x})$ up to equivalence modulo $T$.

# Counting types

- We fully characterised $\aleph_0$-categoricity in terms of $|S_n(T)|$ (or $|S_n(M;A)|$ for finite sets $A$).
- What are the possibilities of $|S_n(T)|$? Notice that $|S_n(T)| \leq 2^{\aleph_0}$ and we have at most countable isolated types.
- If $|S_n(T)| > \aleph_0$ then we have uncountable many countable models of $T$.

*Theorem 6.10. Assume $T$ is a complete theory with infinite models in a countable language $L$. If $|S_n(T)| > \aleph_0$ then $|S_n(T)| = 2^{\aleph_0}$. Actually, $|S_n(T)| \leq \aleph_0$ if and only if isolated types are dense in $S_n(T)$.*

Proof. Since $|S_n(T)| > \aleph_0$ then there is $\phi$ such that $|[\phi]| > \aleph_0$.
Claim: For every such $\phi$ there is $\psi$ such that $|[\phi \wedge \neg\psi]| > \aleph_0$ and $|[\phi \wedge \psi]| > \aleph_0$.
If not show that $p := \{\psi : |[\psi \wedge \psi]| > \aleph_0\}$ is consistent (i.e. finitely consistent).
This implies that $S_n(T) = \bigcup_{\psi \notin p}[\phi \wedge \psi] \cup \{p\}$; contradiction (with $|S_n(T)| > \aleph_0$).
Now using the proof build the following tree:
Argue each branch is leading us to a type
Different branches give us different types
Number of branches is $2^{\aleph_0}$.




It only remains to show if isolated types are not dense then $|S_n(T)| > \aleph_0$. Try to build a tree by taking basic open sets that do not contain isolated types.

□

# Counting types

Definition 6.11. A theory $T$ is called small if $|S_n(T)| \leq \aleph_0$ for all $n \in \mathbb{N}$.

*Lemma 6.12. A complete countable theory is small if and only if it has countable saturated model.*

Proof. Repeat a similar argument to Theorem but with some changes. Start with a countable model $M_0$. In each step $n+1$ choose $M_{n+1}$ to be a countable model of $\mathrm{Diag}_{el}(M_n) \cup \bigcup_p p(x_p)$ for all 1-types $p$ over all finite sets $A \subseteq M_n$. The desired structure is the union of the elementary chain i.e. $M = \bigcup_{n\in\omega} M_n$. □

*Theorem 6.13 (Vaught). A countable complete theory can not have exactly two countable models.*

Proof. We only need to check it for small theories. Since small it has countable saturated model.Since not $\aleph_0$-categorical there is a non-isolated type that can be omitted in a countable model.That type has a realisation in the saturated model. Take a realisation and consider $Th(M, \bar{a})$.This is not $\aleph_0$-categorical hence has a model that is not saturated.This is not isomorphic to any of the others. □

Conjecture [Vaught's Conjecture] If a complete countable theory has fewer than continuum many countable non-isomorphic models, the number of countable models is at most countable.

# Stability

Exercise 6.14. Theorem 6.10 is valid if we consider $S_n(M; A)$ for $A$ countable.

- This would lead to existence of a Prime model: A model that embeds elementary into every model of $T$ (e.g. $\mathbb{Q}^{alg}$ is a prime model of $\mathrm{ACF}_0$).

Definition 6.15. Suppose $T$ is a complete theory in a countable language and $\kappa$ and infinite cardinal. Then we say $T$ is $\kappa$-stable if $|S_n(M; A)| \leq \kappa$ for all $A \subseteq M \models T$ with $|A| \leq \kappa$.When $\kappa = \aleph_0$ then we say the theory $T$ is $\omega$-stable.

Example 6.16.

- ACF is $\omega$-stable.
- DLO is not $\omega$-stable.

Exercise 6.17. If $T$ is $\omega$-stable, then it is $\kappa$-stable for all infinite $\kappa$.

*Proposition 6.18. If $T$ is $\omega$-stable, then the isolated types in $S_n(M; A)$ are dense. If $M \models T$ and $A \subseteq M$, then there is a prime model $M_0 \prec M$ that contains $A$.*

# Categoricity and Morley's theorem

Suppose $T$ is countable complete theory. Here are examples of structures we have have seen so far

- $\aleph_0$-categorical but not $\kappa$-categorical for all other $\kappa$ (DLO, RG).
- $\kappa$-categorical for all infinite $\kappa$ (infinite set, infinite dim vector space over $\mathbb{F}_p$).
- $\kappa$-categorical in every $\kappa$ for all but $\aleph_0$ (ACF).
- Not categorical for all $\kappa$ (RCF).

This was conjectured by Łoś to be the full picture. It turns out to be true.

*Theorem 6.19 (Morley 1965). Suppose $T$ is a complete theory in a countable language with infinite models. If $T$ is $\kappa$-categorical for some $\kappa \geq \aleph_0$ then it is categorical for all uncountable $\kappa$.*

Examples 6.20.

- Infinite sets
- Divisiable abelian groups
- ACF ...

# Counting types beyond $\omega$-categorical theories

Definition 6.21. Let $T$ be a complete $L$-theory, $A \subseteq M \models T$, $M$ infinite. Let $S(M;A) := \bigcup_n S_n(M;A)$. The stability function of a theory is defined as follows: Given and infinite $\kappa$

$$f_T(\kappa) := \sup\{|S(M;A)| : A \subseteq M \models T, |A| = \kappa\}.$$

- The types $\{x = a\}$ for $a \in A$ are all distinct, hence $|S(M;A)| \geq |A|$.
- Moreover, $|S(M;A)| \leq |L(A)\text{-formulas}| \leq 2^{|L|+\aleph_0+|A|} = 2^{\aleph_0+|A|}$.
- Therefore, $\kappa \leq f_T(\kappa) \leq 2^\kappa$.

Examples 6.22.
- Vector space $V$ over $\mathbb{Q}$: $n$-types $\approx$ affine subspace of $V^n \Rightarrow$ $|S_n(V;A)| = |A| + \aleph_0 \Rightarrow f_{\mathrm{VS}_\mathbb{Q}}(\kappa) = \kappa$.
- $K \models \mathrm{ACF}_0$: $n$-types over $A \approx$ prime ideals with coefficients in $\mathbb{Q}(A) \Rightarrow$ $|S_n(K;A)| = |A| + \aleph_0 \Rightarrow f_{\mathrm{ACF}_0}(\kappa) = \kappa$.
- $\mathbb{Q} \models$ DLO: 1-types over $\mathbb{Q} \approx$ Dedekind cuts + infinite, infinitesimals $\Rightarrow$ $|S_1(\mathbb{Q};\mathbb{Q})| = 2^{\aleph_0}$. In general, $f_{\mathrm{DLO}}(\kappa) = \mathrm{ded}(\kappa)$, where
  $$\mathrm{ded}(\kappa) := \sup\{|A| : A \text{ linear order with a dense subset of size } \kappa\}.$$

# Consequences of stability

*Theorem 6.23 (Keisler; Shelah '78). The stability function of $f_T$ must be one of the following values for every infinite $\kappa$:*

$$\kappa, \quad \kappa + 2^{\aleph_0}, \quad \kappa^{\aleph_0}, \quad \text{ded}(\kappa), \quad \text{ded}(\kappa)^{\aleph_0}, \quad 2^{\kappa}.$$

Exercise 6.24.

Theory $T$ is $\omega$-stable if and only if $f_T(\kappa) = \kappa$.

Definition 6.25. Theory $T$ is superstable if $f_T(\kappa) \leq \kappa + 2^{\aleph_0}$. We say $T$ is stable if $f_T(\kappa) \leq \kappa^{\aleph_0}$ (for all infinite $\kappa$).

Definition 6.26. Theory $T$ has the order property if there are $\phi(\overline{x}, \overline{y})$, $M \models T$, and sequences $(a_i)_{i\in\omega}$, $(b_j)_{j\in\omega}$ in $M$ such that $M \models \phi(a_i, b_j) \Leftrightarrow i \leq j$ (i.e., $\phi$ encodes an order).

*Proposition 6.27. $T$ has the order property $\Leftrightarrow f_T(\kappa) > \kappa^{\aleph_0}$, hence $T$ unstable.*

Sketch. ($\Rightarrow$) By compactness, use $\phi$ to encode DLO of size $\kappa$: $f_T(\kappa) \geq \text{ded}(\kappa)$.
($\Leftarrow$) Not order property $\Rightarrow$ types are "definable", hence $f_T(\kappa) \leq \kappa^{\aleph_0}$. □

- There are some interesting geometric consequence of stability; some nice *dimension theory* called Morley rank.

# Classification; Map of the universe

Check the map of the universe: http://forkinganddividing.com

## Exercises 6.28.

1. Prove that $f_T(\kappa) = \sup\{|S_1(M; A)| : A \subseteq M \models T, |A| = \kappa\}$. This would also imply that when we check $\kappa$-stablity we only need to check 1-types.
   Hint: if $b_0 b_1 \models p(x_0 x_1/A)$, then $p$ is determined by $\mathrm{tp}(b_0/A)$ and $\mathrm{tp}(b_1/Ab_0)$.
2. Let $M \models \mathrm{DLO}$. Show that the space of 1-types $S_1(M; M)$ can be ordered in a suitable way so that $M$ embeds into $S_1(M; M)$ as a dense subset (where by dense, we mean that for all $x < y \in S_1(M; M)$, there is $z \in M$ such that $x \leq z \leq y$). Deduce that $|S_1(M; M)| \leq \mathrm{ded}(\kappa)$.
   (Harder!) Prove $\sup_{|M|=\kappa} |S_1(M; M)| = \mathrm{ded}(\kappa)$.
3. Prove that $f_{\mathrm{RCF}}(\kappa) = \mathrm{ded}(\kappa)$.
   Hint: Take $\mathbb{R} \models \mathrm{RCF}$ Prove that every definable subset of $\mathbb{R}$ is definable using only $<$ (hence it's a finite union of points and intervals). Then use Q1 and Q2.
4. Let $L = \{E\}$ be the language with one binary relation symbol $E$. Let $T$ be the theory stating that $E$ is an equivalence relation with infinitely many classes, each one infinite in size. $T$ has quantifier elimination (try to prove it when revising the course).
   a. Computer the size of $S_1(M; A)$ for $A \subseteq M \models T$.
   b. Using Q1, deduce that $T$ is $\omega$-stable.