An excursion into Model Theory and its applications; Lecture 5

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LMS online lecture series – Fall 2020

Lecture 5

Types and saturated models

Types, *informal*

Consider $L_{\mathrm{oring}} = \{<, 0, 1, +, \cdot\}$.

- Let $\mathbb{R}^* := \mathbb{R}^{\mathbb{N}}/U$ where U is a non-principal ultrafilter ($\mathbb{R}^*$ called non-standard extension of $\mathbb{R}$).
- By Łoś's Theorem we have $\mathbb{R} \prec \mathbb{R}^*$.
- In $\mathbb{R}^*$ there exist infinitesimal positive numbers:
  There exists $a \in \mathbb{R}^*$ such that $0 < a < \frac{1}{n}$ for every $n \in \mathbb{N}$
  (recall the element $\epsilon = [(1, \frac{1}{2}, \frac{1}{3}, \ldots)]$).

Some proofs in $\mathbb{R}$ can be simplified by working in $\mathbb{R}^*$, using infinitesimals.

- To understand a structure M, it can be useful to pass to some $M^* \succ M$ since M^* contains "idealised elements".
- The *idealised elements* in M^* are those that are described by (infinitely many) $L(M)$-formulas.

In $\mathbb{R}$ (in L_{oring}) for every $n \in \mathbb{N}$ consider the formula $\phi_n(x) : 0 < x < \frac{1}{n}$.An infinitesimal element in $\mathbb{R}^*$ is a realisation of $\Sigma(x) := \{\phi_n(x) : n \in \mathbb{N}\}$.Sets of formulas like $\Sigma(x)$ are called *types*.

Types, formal

Fix a language L, an L-theory T and $n \in \mathbb{N}$. Below, all tuples $\overline{x}$, $\overline{a}$ are n-tuples.

Definition 5.1. Consider a set $\Sigma(\bar{x})$ of L-formulas.

- A realisation of $\Sigma(\overline{x})$ in some $M \models T$ is a tuple $\overline{a} \in M^n$ such that $M \models \phi(\overline{a})$ for all $\phi(\overline{x}) \in \Sigma(\overline{x})$. Notation: $M \models \Sigma(\bar{x})$.
- $\Sigma(\overline{x})$ is consistent (with T) if $\Sigma(\bar{x})$ has a realisation.
- $\Sigma(\bar{x})$ is complete (modulo T) if for every $\psi(\overline{x})$, either $\Sigma(\bar{x}) \vdash \psi$ or $\Sigma(\bar{x}) \vdash \neg\psi$.
- A complete and consistent set of L-formulas $\Sigma(\bar{x})$ (in n-variables) is called an (n-)type of T.

Fact 5.2. Every consistent set $\Sigma(\overline{x})$ of formulas is contained in a type (Zorn's Lemma).

Example 5.3.

- In $\mathbb{R}$ (in L_{oring}) the set $\Sigma(x) = \{0 < x < \frac{1}{n} : n \in \mathbb{N}\}$ is a 1-type. (Note that $x < \frac{1}{3}$ is a shorthand for $(1+1+1) \cdot x < 1$).
- Any infinitesimal element of $\mathbb{R}^*$ is a realisation of $\Sigma(x)$.

Exercise 5.4. Prove that $\Sigma(x) = \{0 < x < \frac{1}{n} : n \in \mathbb{N}\}$ is complete modulo RCF.

Consistency of types

Proposition 5.5. A set of formulas $\Sigma(\overline{x})$ is consistent if and only if it is finitely consistent (with T).

Proof.

- Set $L' := L \cup C_n$, where C_n is set of size n of new constants.
- Set $T' := T \cup \Sigma(\bar{c})$, where $\Sigma(\overline{c}) := \{\phi(\overline{c}) : \phi(\overline{x}) \in \Sigma(\overline{x})\}$.
- When $\Sigma(\bar{x})$ is finitely consistent, then T' is finitely consistent. (A realisation $M \models \Sigma(\overline{a})$ yields a model of $T \cup \Sigma(\overline{c})$ by setting $\overline{c}^M = \overline{a}$.)
- Compactness Theorem implies T' is consistent, i.e., there exists $M' \models T'$. Hence $\Sigma(\bar{x})$ is consistent (with T). □

Example 5.6.

- Consider again $T = \mathrm{RCF}$ (in L_{oring}) and $\Sigma(x) = \{0 < x < \frac{1}{n} : n \in \mathbb{N}\}$.
- Any finite subset of $\Sigma(x)$ has a realization $a \in \mathbb{R}$: just take a small enough.
- Thus $\Sigma(x)$ is consistent.

Completeness of types

- Proving completeness of sets of formulas is often hard: normally, we need to rely on good quantifier elimination results (compare Exercise 5.4).

An easy way to obtain types:

Definition 5.7. Given a structure M and $\overline{a} \in M^n$, set $\mathrm{tp}(\overline{a}) := \{\phi(\overline{x}) \mid M \models \phi(\overline{a})\}$ (the type of $\overline{a}$).

Exercise 5.8. Prove that every type of a theory T is the type of some $\overline{a}$ for some $\overline{a} \in M^n$ in some $M \models T$.

Exercise 5.9. Let $M_1, M_2 \models T$ and $\bar{a}_i \in M_i^n$. The type $tp(\bar{a}_i)$ can be identified with the complete $L(C_n)$-theory $Th(M, \bar{a}_i)$. In particular, $tp(\bar{a}_1) = tp(\bar{a}_2)$ if and only if $(M_1, \bar{a}_1) \equiv_{L(C_n)} (M_2, \bar{a}_2)$

Exercise 5.10. Write some types of $\mathrm{Th}(\mathbb{Z})$ in the language $L = \{+\}$. Find one that is *not* realised in $\mathbb{Z}$.

Isolated types

We have seen how to realise a type.

Definition 5.11. Consider a set $\Sigma(\bar{x})$ of L-formulas and T an L-theory. A model $M \models T$ omits $\Sigma(\bar{x})$ if $\Sigma(\bar{x})$ is not realised in M.

Example 5.12. Again in L_{oring} and $T = RCF$. Then $\mathbb{R}$ omits $\Sigma(x) = \{0 < x < \frac{1}{n} : n \in \mathbb{N}\}$.

Definition 5.13. Suppose $\Sigma(\bar{x})$ is a consistent set of $\mathcal{L}$-formulas. An L-formula $\psi(\bar{x})$ isolates $\Sigma(\bar{x})$ if $T \cup \{\psi(\bar{x})\}$ is consistent and for all $\phi(\bar{x}) \in \Sigma(\bar{x})$

$$T \vdash \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x})).$$

- If $\psi(\bar{x})$ isolates $\Sigma(\bar{x})$ then a model of T that realises $\psi(\bar{x})$ realises also $\Sigma(\bar{x})$.
- If T is a complete theory every isolated $\Sigma(\bar{x})$ is realised in T (in every model).

Omitting Types Theorem

Theorem 5.14. Let T be an L-theory in a countable language L and $\Sigma(\bar{x})$ a consistent set of L-formulas. If $\Sigma(\bar{x})$ is not isolated then T has model which omits $\Sigma(\bar{x})$.

Proof. Assume $\bar{x}$ is 1-tuple. Let $L(C) := L \cup C$ where C is a countable set of new constants. Goal: Find T^* an extension of T with the following properties:

1. For every $L(C)$-formulas $\psi(x)$ there exists $c \in C$ such that $\exists x.\psi(x) \to \psi(c) \in T^*$.
2. For every $c \in C$ there is $\theta(\bar{x}) \in \Sigma(\bar{x})$ such that $\neg\theta(c) \in T^*$.

If T^* is consistent then we are done! Take a countable model that contain only the constants (use a Downward Löwenheim-Skolem and Tarski's test)
Construct T^* inductively. Enumerate C and $L(C)$-formulas $\{\psi_i : i \in \omega\}$.
Construct $T =: T_0 \subseteq T_1 \subseteq \ldots$ of consistent extension of T by finite set of $L(C)$-sentences satisfying 1 and 2.
If T_{2k} is constructed pick $c \in C$ that does not occur in $T_{2k} \cup \{\psi_i(\bar{x})\}$ and consider $T_{2k+1} := T_{2k} \cup \{\exists x.\psi(x) \to \psi(c)\}$. Suppose T_{2k+1} is constructed. Note that $T_{2k+1} = T \cup \{\delta(c_k, \bar{c})\}$. The formula $\exists \bar{y}\delta(x, \bar{y})$ does not isolate $\Sigma(x)$. Hence there exists $\theta(x) \in \Sigma(x)$ such that $T \cup \{\exists \bar{y}\delta(x, \bar{y}) \wedge \neg\theta(x)\}$ is consistent. Thus $T_{2k+2} := T_{2k+1} \cup \{\neg\theta(c_k)\}$ is consistent. □

Types over a set

Definition 5.15. Fix an L-structure M and $B \subseteq M$.

- A type over B is a type in the language $L(B)$ of the $L(B)$-theory $\mathrm{Th}(M)$.
- For $M' \succ M$ and $\overline{a} \in (M')^n$ the type of $\overline{a}$ over B $\mathrm{tp}(\overline{a}/B)$ is the type of $\overline{a}$ in the language $L(B)$.

Example 5.16. Let $L = \{<\}$ and $T = \mathrm{DLO}$.

- The only 1-type of $\{x = x\}$ (that's essentially the only consistent formula we can write!). The only 2-types are $\{x = y\}, \{x < y\}$ and $\{y < x\}$.
- Any real number $r \in \mathbb{R}$ yields a type $p_r(x) := \mathrm{tp}(r/\mathbb{Q})$.
  Example: $p_\pi(x)$ contains $3 < x$, $3.1 < x$, ..., $x < 4$, $x < 3.2$, ...
- For $r < r'$, we have $p_r \neq p_{r'}$, since for $q \in \mathbb{Q}$ with $r < q < r'$, we have $(q < x) \in p_{r'}$ but $(q < x) \notin p_r$.
- But there exist even more 1-types over $\mathbb{Q}$, e.g. $\{0 < x\} \cup \{x < q \mid q \in \mathbb{Q}, q > 0\}$. Exercise: make a list of *all* 1-types over $\mathbb{Q}$.

Exercise 5.17. Let $K \models \mathrm{ACF}$. Show that 1-types over K correspond to prime ideals of $K[X]$. (Hint: map $I \mapsto \{P(x) = 0 \mid P \in I\} \cup \{Q(x) \neq 0 \mid Q \notin I\}$.) Generalise this to 1-types over L when L is a sub field of K.

Saturated models

Definition 5.18. A structure M is κ-saturated if for every $A \subseteq M$ of size $|A| < \kappa$, every type over A is realised in M; it is saturated if it is $|M|$-saturated.

Examples 5.19.

- $\mathbb{Q}^{alg} \models$ ACF is not $\aleph_0$-saturated: $\{P(x) \neq 0 \mid P \in \mathbb{Z}[X] \setminus \{0\}\}$ is not realised (what ideal corresponds to this type)?
- The algebraic closure of $\mathbb{Q}(T_1, T_2, \ldots)$ is saturated.
- $\mathbb{C} \models$ ACF is saturated.

Proposition 5.20. Suppose M is κ-saturated, $N \equiv M$ and cardinality of $|N| \leq \kappa$. Then there is an elementary embedding of N into M.

Proof. Enumerate elements of $N := \{n_i : i \in I\}$ where $|I| \leq \kappa$. Consider $\Sigma = \{\phi(x_i) : N \models \phi(n_i), i \in I\}$. Since $N \equiv M$ then Σ is finitely satisfiable in M. Since M is κ-saturated then Σ is realised in M. Now define a function between N and satisfaction of Σ and show it is an elementary map. □

Exercise 5.21. Suppose M and N are two saturated structures and $N \equiv M$. Then M and N are isomorphic.

Saturated models

Example 5.22. Let $L = \{<\}$. Recall $\mathbb{Q} \prec \mathbb{R}$. Then:

- $\mathbb{Q}$ is saturated ($\aleph_0$-saturated);
- $\mathbb{Q}$ is not $\aleph_1$-saturated: $\mathrm{tp}(\pi/\mathbb{Q})$ is not realised;
- $\mathbb{R}$ is not $\aleph_1$-saturated: $\{0 < x < q \mid q \in \mathbb{Q}^{>0}\}$ is not realised.

An $\aleph_1$-saturated extension of $\mathbb{Q}$ will contain infinite points, infinitesimals, ...
It must have size $\geq 2^{\aleph_0}$ (as that's the number of types over $\mathbb{Q}$).

Exercise 5.23. Suppose U is a non-principal ultrafilter on $\mathbb{N}$. Assume M_i for $i \in \mathbb{N}$ is a family of L-structures. Show $\prod_{i \in \mathbb{N}} M_i / U$ is $\aleph_1$-saturated.

Proposition 5.24. For all M and all infinite κ, there is a κ-saturated $M' \succ M$.

Proof. We obtain the desired structure as union of an elementary chain of length κ^+ of structures (κ^+ is the first cardinal bigger than κ). Let $M_0 = M$ and in a successor step $\alpha = \beta + 1$ choose M_α to be the model of $\mathrm{Diag}_{el}(M_\beta) \cup \bigcup_p p(x_p)$ for all 1-types p over sets $A \subseteq M$ such that $|A| < \kappa$. In the limit ordinal $M_\alpha := \bigcup_{\beta < \alpha} M_\beta$. Now let $M' = \bigcup_{\alpha < \kappa^+} M_\alpha$. Show that M' is κ-saturated (use the fact that κ^+ is a regular cardinal). □

Exercises 5.25.

1. Let $L = L_{\text{ring}}$, $K \models \text{ACF}$. Let $B \subseteq K$ be a subset. Let K_0 be the subfield of K generated by B. Consider the type $\text{tp}(a/B)$ of an element $a \in K$. Prove:
    1. If a is algebraic over K_0, then there exist only finitely many $a' \in K$ with $\text{tp}(a'/B) = \text{tp}(a/B)$. What ideal corresponds to $\text{tp}(a/B)$?
    2. If a is not algebraic over K_0, then for any a' that is not algebraic over K_0, we have $\text{tp}(a'/B) = \text{tp}(a/B)$. What ideal corresponds to $\text{tp}(a/B)$?
    3. What is the unique non-realised type over K? Why?
2. Give a list of all 3-types of DLO.
3. Consider $\mathbb{R} \models \text{DLO}$. Give a list of all 1-types over $\mathbb{Q}$ and over $\mathbb{R}$ in $L = \{<\}$.
4. Call a graph random if for every disjoint finite sets of vertices A, B, there is a vertex v connected to every point of A but to no point of B. Let G, H be random graphs.
    1. Axiomatise the theory RG of random graphs (with $L = \{E\}$).
    2. Prove that the family $\mathcal{I}$ of all partial isomorphisms $G \to H$ with finite domains, i.e., the isomorphisms between finite (induced) subgraphs of G and H, has the back-and-forth property.
    3. Hence RG has quantifier elimination. List all 1-types over G.