An excursion into Model Theory and its applications; Lecture 4

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Lecture 4

Löwenheim–Skolem Theorem

Review

[Definition 1.9] Map $h : M \to N$ between two L-structures is a homomorphism if

1. $h(c^M) = c^N$ for all constant symbols $c \in L$;
2. $h(f^M(a_1, \ldots, a_n)) = f^N(h(a_1), \ldots, h(a_n))$ for all function symbols $f \in L$ and $a_1, \ldots, a_n \in M$;
3. if $(a_1, \ldots, a_m) \in R^M$ then $(h(a_1), \ldots, h(a_m)) \in R^N$ for all relation symbols $R \in L$ and $a_1, ..., a_m \in M$.

Map h is an L-embedding it is an injective homomorphism and in addition $(a_1, \ldots, a_m) \in R^M \Leftrightarrow (h(a_1), \ldots, h(a_m)) \in R^N$ for all $a_1, ..., a_m \in M$.

Fact 4.1. An L-homomorphism $h : M \to N$ preserves atomic formulas i.e. for every atomic formula ϕ

$$M \models \phi(a_1, \ldots, a_m) \Rightarrow N \models \phi(h(a_1), \ldots, h(a_m)).$$

A homomorphism is an embedding if and only if it preservers negation of atomic formulas or in other words for every atomic formula ϕ

$$M \models \phi(a_1, \ldots, a_m) \Leftrightarrow N \models \phi(h(a_1), \ldots, h(a_m)).$$

Elementary embedding and elementary substructures

Exercise 4.2. An embedding preserves formulas of the form $\exists x.\phi(x)$ where ϕ is a quantifier-free formula.

Definition 4.3. Suppose M and N are two L-structures. A map $h : M \to N$ is an elementary embedding if it *preserves* all the first order L-formulas i.e. for all $a_1, \ldots, a_m \in M$

$$M \models \phi(a_1, \ldots, a_m) \Leftrightarrow N \models \phi(h(a_1), \ldots, h(a_m)).$$

A substructure M_0 of M is called an elementary substructure if the inclusion map is elementary. Denoted by $M_0 \prec M$ or $M_0 \prec_L M$. In this case M is called elementary extension of M_0.

Fact 4.4. Suppose M_0 is an elementary substructure of M if and only if for every L-formula $\phi(\overline{x})$, $\phi(M_0) = \phi(M) \cap M_0^n$.

Example 4.5.

- In $L = \emptyset$ for infinite sets every substructure is elementary.
- $2\mathbb{Z} \not\prec_{L_{\mathrm{oag}}} \mathbb{Z}$, because $\phi(2\mathbb{Z}) \neq \phi(\mathbb{Z}) \cap 2\mathbb{Z}$ for $\phi : \exists y.y + y = x$.

Application: Hilbert's Nullstellensatz

Proposition 4.6. If $K_1 \subseteq K_2$ are algebraically closed fields, then $K_1 \prec_{L_{\mathrm{ring}}} K_2$.

Proof. Goal: $K_1 \models \phi(\overline{a}) \Leftrightarrow K_2 \models \phi(\overline{a})$ for every L_{ring}-formula $\phi(\overline{x})$ and $\overline{a} \in K_1$. By q.e., we may assume that $\phi(\overline{x})$ has no quantifiers. Then the goal is trivially true. □

Theorem 4.7 (Weak Hilbert's Nullstellensatz, 1893). Given an algebraically closed field K and polynomials $f_1, \ldots, f_k \in K[\overline{x}]$ with $1 \notin I := (f_1, \ldots, f_k)$, there exists $\overline{a} \in K^n$ such that $f_1(\overline{a}) = \ldots = f_k(\overline{a}) = 0$.

Proof. Consider the $L_{\mathrm{ring}}(K)$-sentence σ: $\exists \bar{x}.(f_1(\overline{x}) = 0 \wedge \ldots \wedge f_k(\bar{x}) = 0)$.

- Instead of proving $K \models \sigma$, it suffices to prove $K' \models \sigma$ for any $K' \succ K$...
- ... i.e., for any algebraically closed $K' \supseteq K$.
- Choose a maximal ideal $M \supseteq I$.
- Take the field extension $K'' := K[\overline{x}]/M \supseteq K$. Set $K' := (K'')^{alg}$.
- Write $q : K[\overline{x}] \to K''$ for the quotient map.
- Then for all i, $f_i(q(x_1), \ldots, q(x_n)) = q(f_i(x_1, \ldots, x_n)) = 0$ (since $f_i \in M$).
- Thus $K' \models \sigma$, namely with $\overline{a} = (q(x_1), \ldots, q(x_n))$. □

Model completeness

Definition 4.8. An L-theory T is called model complete if for all models of N_1, N_2 of T we have

$$N_1 \subseteq N_2 \Rightarrow N_1 \prec N_2.$$

Exercise 4.9. All theories with q.e. are model complete. In particular, if M is a model of a T with q.e. every substructure of M that is a model of T is elementary.

Robinson's Test: T is model complete if and only if every formula is, modulo T, equivalent to a universal formula (i.e. of the form $\forall x.\phi(x)$ where ϕ is q.f.)

Tarski's test

To check that a substructure is an elementary substructure, use the following.

Proposition 4.10 (Tarski's test). *Given L-structures $M_0 \subseteq M$, TFAE:*

1. *$M_0 \prec M$;*
2. *For every $L(M_0)$-formula $\phi(x)$, we have:*
   there is $a \in M$ with $M \models \phi(a) \quad \Leftrightarrow \quad$ there is $a_0 \in M_0$ with $M \models \phi(a_0)$.

Proof of (1) $\Rightarrow$ (2). $M \models \phi(a) \quad \Rightarrow \quad M \models \exists x.\phi(x) \quad \overset{(1)}{\Rightarrow} \quad M_0 \models \exists x.\phi(x)$

$\Rightarrow \quad M_0 \models \phi(a_0)$ for some $a_0 \in M_0 \quad \overset{(1)}{\Rightarrow} \quad M \models \phi(a_0)$. □

Proof of (2) $\Rightarrow$ (1). Given σ an $L(M_0)$-sentence, prove $M_0 \models \sigma \Leftrightarrow M \models \sigma$.

- We do an induction on the complexity of σ.
- Only the case $\sigma : \exists x.\psi(x)$ is non-trivial. In that case:
  - By definition: $M \models \sigma \iff$ there is $a \in M$ s.t. $M \models \psi(a)$;
  - By the assumption: $\iff$ there is $a_0 \in M_0$ s.t. $M \models \psi(a_0)$;
  - By induction: $\iff$ there is $a_0 \in M_0$ s.t. $M_0 \models \psi(a_0)$;
  - $\iff M_0 \models \sigma$. □

Löwenheim–Skolem theorem

Given an L-structure M and $A \subseteq M$. We do the following

1. Let Σ_0 be the collection of all $L(A)$-formulas ϕ that $M \models \exists x.\phi(x)$.
2. Let $A_1 \supseteq A$ with witnesses for each formula in Σ_0 (cardinality of A_1?).
3. Repeat similar procedure for A_i's inductively and construct $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.
4. $M_0 := \bigcup_{i \in \omega} A_i$. Then $M_0 \prec M$ where $|M_0| \leq \max\{\aleph_0, |L|, |A|\}$.

Theorem 4.11 (Löwenheim–Skolem). Let M be an infinite L-structure and $A \subseteq M$. Assume κ is an infinite cardinal.

- *Downwards: If $\max\{|L|, \aleph_0, |A|\} \leq \kappa \leq |M|$, there exists $A \subseteq M_0 \prec M$ with $|M_0| = \kappa$.*
- *Upwards: If $\kappa \geq \max\{|L|, |M|\}$, then there exists $M' \succ M$ with $|M'| = \kappa$.*

Proof of upwards.

Let $L^\star := L(M) \cup C$, where C are κ-many new constants.

- Consider the $L^\star$-theory $T := \mathrm{Diag}(M) \cup \{\text{all constants from } C \text{ are different}\}$.
- T is finitely consistent. By compactness theorem, it has a model $M^* \models T$; in particular $M^* \succ_L M$, and all constants being different implies $|M^*| \geq \kappa$.
- Use "downwards", to find $M' \prec_{L(M)} M^*$ with $|M'| = \kappa$ and get $M \prec_L M'$. □

Consequences of Löwenheim–Skolem

Corollary 4.12. A theory that has an infinite model has models in every cardinality $\kappa \geq \max\{|L|, \aleph_0\}$.

Examples 4.13.

$M = \mathbb{R}$ as L_{oring}-structure: $\mathrm{Th}(M) = \mathrm{RCF}$ has other models.

$M = \mathbb{C}$ as L_{ring}-structure: $\mathrm{Th}(M) = \mathrm{ACF}_0$ has other models.

For any infinite cardinal κ, there is $K \models \mathrm{ACF}_0$ with $|K| = \kappa$.

- Direct proof: take $\mathbb{Q}$; adjoin κ many transcendental elements; take the algebraic closure of that. The resulting field has cardidinality κ.

- One can also prove: Given M_1, M_2 with $M_1 \equiv M_2$, there exists M with $M_1 \prec M$, $M_2 \prec M$.
- Common application: Suppose T is a complete theory. Then one can work in one huge $M \models T$ which contains all the $M_i \models T$ one might ever be interested in as elementary substructures. (Such a huge M is called a monster model.)

Consequences of Löwenheim–Skolem

[Exercise 2.11] If two models are isomorphic they are elementary equivalent.
The converse is not true at least for cardinality issues.
What if the structures have the same cardinality and elementary equivalent?
(Cantor) Every countable dense linear order without endpoints is isomorphic to the rational numbers (Theorem 2.28. DLO is $\aleph_0$-categorical).

Exercise 4.14. If two structures are Back and Forth equivalent and countable then they are isomorphic.

[Definition 2.21] Let $\kappa \geq \max\{|L|, \aleph_0\}$. An L-theory T is called κ-categorical if all models of T of cardinality κ are isomorphic.

[Lecture 2 - Vaught's test] Suppose T is a theory with no finite models and it is κ-categorical for some infinite $\kappa \geq |L|$. Then T is complete.

Proof. We show every two model M, N of T are elementary equivalent. Since M and N are infinite then $Th(M)$ and $Th(N)$ have a model M' and N' of cardinality κ. Note that $M', N' \models T$. Hence by κ-categoricity

$$M \equiv M' \cong N' \equiv N.$$

□

Historical remarks and discussion

- Compactness and Upward Löwenheim–Skolem theorem are consequences of Gödels Completeness theorem.
- By a theorem by Per Lindström the first order logic is the only *abstract logic* that satisfies Countable Compactness Theorem and Downward Löwenheim–Skolem Property.
- Skolem's paradox. Every first order axiomatizing of set theory has a countable model ... axioms that might seem to imply there are uncountable sets.

Exercise 4.15.

- The 'connectedness' of a graph in L_{graph} is not a first-order expressible.
- The 'Archimedean property' of an ordered field in L_{oring} is not first-order expressible. Recall that an ordered F is Archimedean if 'for all positive $x \in F$ there exists $n \in \mathbb{N}$ such that $nx > 1$'.

- (Dedekind) In the language $\{s, 0\}$ there a *second order sentence* axiomatisation of the natural numbers that is 'categorical'.

What is next?

- Types, space of types.
- Saturated models.
- Omitting type theorem.

Exercises 4.16.

1. Let L be a language and let M_1 and M_2 be L-structures with $M_1 \equiv M_2$. Prove that there exists an L-structure N which has elementary substructures N_1 and N_2 with N_i isomorphic to M_i, i.e., $M_1 \cong N_1 \prec N \succ N_2 \cong M_2$.
2. Provide an example of two non-isomorphic L-structures A_1 and A_2 (in your favourite language L) where both A_1 and A_2 are countable, A_1 is a substructure of A_2 and they are elementarily equivalent.
3. Find L-structures A and B which are not elementarily equivalent and such that A is isomorphic to a substructure of B and B is isomorphic to a substructure of A.
4. Consider the L-structure $N =< \mathbb{N}; f >$ where f is a binary function symbol with $f^N(a, b) := a + b$ for all $a, b \in N$. First for every prime number p find a formula $\phi_p(v)$ such that $N \models \phi_p(n)$ if and only if p divides n. Suppose P and Q are two disjoint subsets of prime numbers. Show that there is a countable model M of $Th(N)$ where there is $m \in M$ such that all elements of P divide m but none of the elements of Q divide m.
5. Let $L = \{f, c\}$ be a language with a unary function symbol f and a constant symbol c. Write an L-theory T stating that f is injective, that the image is everything except for c, and that f has *no cycles* ($f \circ \ldots \circ f(x) \neq x$). Prove that T is uncountably categorical: for every cardinal $\kappa > \aleph_0$, all models of T of cardinality κ are isomorphic. Deduce that T is complete.