An Excursion Into Model Theory and Its Applications; Lecture 3

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Lecture 3

Ultraproducts and the Compactness Theorem

Filters

Let I be a set. An (ultra)filter on I is a collection of 'large' subsets of I:

Definition 3.1. A filter on I is a subset F of the powerset $P(I)$ satisfying:

1. $I \in F$, $\emptyset \notin F$ (the whole set is large, the empty set is not large).
2. If $A \in F$ and $A \subseteq B$, then $B \in F$ (any set containing a large set is large).
3. If $A, B \in F$, then $A \cap B \in F$ (intersection of finitely many large sets is large).

- Both A and $A^c = I \setminus A$ cannot belong to a same filter on I.
- Filters have the finite intersection property (fip):

$$\text{If } A_1, \ldots, A_n \in F, \text{ then } \bigcap_{j=1}^{n} A_j \neq \emptyset.$$

- Any non-empty subset $S \subseteq P(I)$ with the fip has a minimal filter containing it, the filter generated by S.

Examples 3.2.

- The trivial filter $F = \{I\}$.
- The principal filter generated by $i \in I$, i.e., $F = \{A \subseteq I : i \in A\}$.
- The Fréchet filter on infinite I: $F = \{A \subseteq I : A^c \text{ is finite}\}$.

Ultrafilters

Definition 3.3. Let U be a filter on I. Then

1. U is an ultrafilter if for all $A \subseteq I$ either $A \in U$ or $A^c \in U$.
2. U is a prime filter if for any $A, B \subseteq I$ with $A \cup B \in U$, either $A \in U$ or $B \in U$.
3. U is a maximal filter if every filter F including U coincides with U.

Proposition 3.4. For a filter U on I, the definitions above are equivalent.

$1 \Rightarrow 2$: Assume $A, B \notin U$. Then $A^c, B^c \in U$ and so $(A \cup B)^c \in U$. Hence $A \cup B \notin U$.
$2 \Rightarrow 3$: Assume $A \notin U$. Then $U \cup \{A\}$ does not have the fip, that is, U is maximal.
$3 \Rightarrow 1$: Assume $A \notin U$. Then $U \cup \{A\}$ does not have the fip. Thus, there is $B \in U$ with $B \cap A = \emptyset$, i.e. $B \subseteq A^c$.

- Principal filters are ultrafilters. If I is finite, any ultrafilter on I is principal.
- The trivial filter and the Fréchet filter are not ultrafilters.
- So are there any non-principal ultrafilters?

Lemma 3.5 (Ultrafilter Lemma). Every filter F is contained in an ultrafilter.

Proof. Zorn's Lemma on the poset of filters on I containing F. □

Corollary 3.6. Any subset $S \subseteq P(I)$ with the fip is contained in an ultrafilter.

Ultraproducts

Definition 3.7. Fix a language L. Let I be a set, U be an ultrafilter on I, and $\{M_i : i \in I\}$ be a family of L-structures. The ultraproduct of the M_i's over U is the L-structure $M = \prod_{i \in I} M_i / U$ defined as follows:

- The domain of M is the quotient of $\prod_{i \in I} M_i$ by the equivalence relation $\sim$ defined by $(a_i) \sim (b_i)$ if they agree on a large set, i.e., if $\{i \in I : a_i = b_i\} \in U$.

Denote by $[(a_i)]$ the equivalence class of the element (a_i) w.r.t the equivalence relation $\sim$. The structure on M is simply the quotient structure, that is:

- If c is a constant symbol of L, $c^M = [(c^{M_i})]$.
- If R is an n-ary relation symbol of L, $([(a_{i,1})], \ldots, [(a_{i,n})]) \in R^M \Leftrightarrow \{i \in I : (a_{i,1}, \ldots, a_{i,n}) \in R^{M_i}\} \in U$.
- If f is an m-ary function symbol of L, $f^M([(a_{i,1})] \ldots, [(a_{i,m})]) = [(f^{M_i}(a_{i,1}, \ldots, a_{i,m}))]$.

Exercise 3.8. Check that $\sim$ is an equivalence relation and the interpretations of f and R are independent of the choice of representative for each equivalence class.

Definition 3.9. If all the L-structures M_i are equal to an L-structure M, then the L-structure $\prod_{i \in I} M_i / U = M^I / U$ an called ultrapower of the M_i's over U.

Łos's Theorem

The ultraproduct contruction is done with the goal in mind that $\prod_{i\in I} M_i/U$ must satisfy any first-order property that holds for almost all of the structures M_i:

Theorem 3.10 (Łos's Theorem). *Fix a language L. Let I be a set, U an ultrafilter on I and $\{M_i : i \in I\}$ a family of L-structures. Let $\phi(\bar{x})$ be an L-formula and let $\overline{[(a_i)]}$ be a tuple of elements from the ultraproduct $\prod_{i\in I} M_i/U$. Then*

$$\prod_{i\in I} M_i/U \models \phi(\overline{[(a_i)]}) \Leftrightarrow \{i \in I : M_i \models \phi(\overline{a_i})\} \in U$$

.

Exercise 3.11. Prove Łos's Theorem by induction on formulas. What goes wrong in the proof if U is assumed to be a filter instead of an ultrafilter?

We have: if $\{M_i : i \in I\}$ is a family of models of a theory T, then $\prod_{i\in I} M_i/U \models T$. E.g, an ultraproduct of fields is a field, and ultraproduct of linear orders is a linear order, etc.

Example 3.12. Let $\{\mathbb{R}_i : i \in \mathbb{N}\}$ be a family of copies of $\mathbb{R}$ and U be a non-principal ultrafilter on $\mathbb{N}$. The ultrapower $\mathbb{R}^{\mathbb{N}}/U$ is a model of $\mathrm{Th}(\mathbb{R})$ which contains infinitesimal elements; consider the element $\epsilon = [(1, \frac{1}{2}, \frac{1}{3}, \ldots)] \in \mathbb{R}^{\mathbb{N}}/U$. For any integer n, $\mathbb{R}^{\mathbb{N}}/U \models \epsilon < \frac{1}{n}$, as the number of factors in which this is true is cofinite, and hence in U.

The Compactness Theorem

Theorem 3.13 (Compactness Theorem, 1st version). For any L-theory T and any L-sentence σ: if $T \vdash \sigma$ then there exists a finite subset $T_0 \subseteq T$ such that $T_0 \vdash \sigma$.

Definition 3.14. T is finitely consistent if every finite subset is consistent.

Theorem 3.15 (Compactness Theorem, 2nd version). If an L-theory T is finitely consistent, then it is consistent.

- Implication 1st $\Rightarrow$ 2nd version: Use $\sigma = \bot$. □
- Implication 2st $\Rightarrow$ 1nd version: $T \vdash \sigma \quad \Rightarrow \quad T \cup \{\neg\sigma\}$ is inconsistent $\Rightarrow \quad T_0 \cup \{\neg\sigma\}$ is inconsistent for some finite $T_0 \quad \Rightarrow \quad T_0 \vdash \sigma$. □

Corollary 3.16. If an L-theory T has no infinite model, then there is some $N \in \mathbb{N}$ such that every model of T has size at most N.

Proof. Suppose there is a sequence of models $M_n \models T$ such that $|M_n| \to \infty$. Then for every finite subset T_0 of

$$T \cup \left\{ \exists x_1 \ldots x_k . \bigwedge_{i \neq j}^{k} x_i \neq x_j \right\}$$

we have $T_0 \models M_n$ for some (all sufficiently large) $n \in \mathbb{N}$. Hence the above set is finitely consistent, so it has a model, and that model must be infinite. □

A proof of the Compactness Theorem

Theorem 3.17. (Compactness Theorem) An L-theory T is consistent if and only it is finitely consistent.

Proof.

$\Rightarrow$ is trivial.

$\Leftarrow$: Let T be an infinite finitely consistent theory. Using ultraproducts, we prove that T is consistent:

- Let I be the set of all finite subsets of T.
- Now have a collection of structures $\{M_i : i \in I\}$ with $M_i \models i$ for all $i \in I$.
- Idea: take the ultraproduct of the M_i's over some ultrafilter U on I so that $\prod_{i \in I} M_i / U$ is a model of T.
- By Łos's Theorem, it is enough to find an ultrafilter U s.t for all $\sigma \in T$, $\{i : M_i \models \sigma\} \in U$.
- $\{i : M_i \models \sigma\} \supseteq \{i : \sigma \in i\}$, since $M_i \models i$.
- Let $A_\sigma = \{i : \sigma \in i\}$. Now $\{A_\sigma : \sigma \in T\}$ has the fip as $\bigcap_{j=0}^{n} A_{\sigma_j}$ contains $\{\sigma_0, \dots, \sigma_n\}$.
- We may now pick U so that $A_\sigma \in U$ for all $\sigma \in T$.

□

An application

We've seen that one can transfer sentences between algebraically closed fields of the same characteristic. Next we see that one can also change the characteristic:

Theorem 3.18. *Let σ be any L_{ring}-sentence. Then TFAE:*

1. $\mathbb{C} \models \sigma$.
2. *If K is an algebraically closed field of characteristic 0, then $K \models \sigma$.*
3. *There exists an integer n, such that if p is a prime number $> n$, then $\mathbb{F}_p^{alg} \models \sigma$*
4. *There exists an integer n, such that if p is a prime number $> n$ and K is an algebraically closed field of characteristic p, then $K \models \sigma$.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) are clear at this point.

- Assume (2): $\mathsf{ACF}_0 = \mathsf{ACF} \cup \{p \neq 0 : p \text{ a prime}\} \vdash \sigma$. Compactness: there are finitely many prime numbers $p_1, \ldots, p_m$ s.t. $\mathsf{ACF} \cup \{p_1 \neq 0, \ldots, p_m \neq 0\} \vdash \sigma$. Take $n > \sup\{p_1, \ldots, p_m\}$. We have (2) $\Rightarrow$ (4).
- Assume (4) and let U be a non-principal ultrafilter on the set P of primes. For each prime p, choose an algebraically closed field K_p of characteristic p. Then $\{p \in P : K_p \models \sigma\} \in U$. Therefore $\prod_{p \in P} K_p / U \models \sigma$. For each p, the set $\{q \in P : q \neq p\}$ is also in the ultrafilter. By Łos's Theorem, $\prod_{p \in P} K_p / U$ is an algebraically closed field of characteristic 0. We have (4) $\Rightarrow$ (2). □

Ax's Theorem

Theorem 3.19 (Ax's Theorem). *Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial map. If f is injective, then it is surjective.*

Proof.

- Theorem 3.19 can be captured using L_{ring}-sentences $\sigma_{n,d}$ which holds in a field F iff every injective polynomial map $F^n \to F^n$ of degree d is surjective.
- Thus, by Theorem 3.18 it suffices to show that for large enough p, any injective polynomial map $f : (\mathbb{F}_p^{alg})^n \to (\mathbb{F}_p^{alg})^n$ is surjective.
- We denote $K = \mathbb{F}_p^{alg}$. Let $\bar{x} \in K^n$. We want to find $\bar{y}$ such that $f(\bar{y}) = \bar{x}$.
- Choose $\mathbb{F}_q \subset K$ which contains $x_1, \ldots, x_n$ and all coefficients in f.
- Since $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is injective it must be surjective as $\mathbb{F}_q$ is finite. Hence there is $\bar{y} \in \mathbb{F}_q^n \subseteq K^n$ with $f(\bar{y}) = \bar{x}$.

□

Finally, a historical remark

- While ultraproducts allow us to give a short and algebraic proof of the Compactness Theorem, this is not at all the original proof!
- Let T be an L-theory and σ an L-sentence. A proof of σ from T is a finite sequence of L-formulas $\phi_1, \ldots, \phi_n$ such that $\phi_n = \sigma$ and, for all $i = 1, \ldots, n$, $\phi_i \in T$ or ϕ_i follows from $\phi_1, \ldots, \phi_{i-1}$ by logical rules. Then $T \vdash \sigma$.

Theorem 3.20 (Gödels Completeness Theorem). *Let T be an L-theory and σ an L-sentence. Then $T \models \sigma$ if and only if $T \vdash \sigma$.*

As proofs are finite, as a corollary of Gödels Completeness Theorem, one gets the Compactness Theorem.

Exercises 3.21. Prove that an ultrafilter U on infinite I is non-principal if and only if it contains the Fréchet filter on I.

1. Let U be the principal ultrafilter on I genereted by $j \in I$. Prove that $\prod_{i \in I} M_i / F \cong M_j$.
2. Use the ultraproduct construction to build a field of characteristic 0 which has exactly one algebraic extension of each degree.
3. Let $\phi(\overline{x}, \overline{y})$ be an L-formula and T be a complete L-theory. Prove that either (a) there is $N \in \mathbb{N}$ such that $\#\phi(M, \overline{b}) \leq N$ for every $M \models T$ and $\overline{b} \in M^{|\overline{y}|}$, or (b) there is $M \models T$ and $\overline{b} \in M^{|\overline{y}|}$ and $\phi(M, \overline{b})$ infinite.
4. Let T be a complete L_{graph}-theory. Prove that if every model of T is connected, then every model has the same finite diameter (maximal distance between any two vertices).
5. Assume the 4-colour theorem: every finite planar graph is 4-colourable. Using compactness, prove that every *infinite* planar graph is 4-colourable. Guide:
   1. Let $L' = L_{\mathrm{graph}} \cup \{C_1, C_2, C_3, C_4\}$, with $C_1, \ldots, C_4$ unary relation symbols.
   2. Write the L'-theory T_C that says that $C_1, \ldots, C_4$ form a *partition* of the domain.
   3. Let Diag$(G) := \mathrm{Th}_{L(G)}(G)$ (the atomic diagram of G). Prove that G is 4-colourable if and only if $\mathrm{Diag}(G) \cup T_C$ is consistent. Then use compactness!
6. Prove that there is an L_{oring}-structure $\mathbb{Z}^* \equiv \mathbb{Z}$ containing an element $a \in \mathbb{Z}^*$ divisible by every $n \in \mathbb{Z} \setminus \{0\}$. Hint: use $L' = L_{\mathrm{oring}} \cup \{c\}$ where c is a new constant symbol. Write an L'-theory expressing that c is divisible by every $n \in \mathbb{Z} \setminus \{0\}$.

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