# An Excursion Into Model Theory and Its Applications; Lecture 2

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Lecture 2

# Theories and quantifier elimination

# Theories

- We usually don't just want $L$-structures for the appropriate $L$, as interpretations could be anything, but we also want to enforce some minimum properties.

Definition 2.1. Fix a language $L$.

- An $L$-theory $T$ is a set of $L$-sentences (usually called axioms).
- A model of a theory $T$ is an $L$-structure $M$ such that $M \models \sigma$ for all $\sigma \in T$.

Example 2.2.

- The theory of abelian groups is the $L_{\mathrm{ag}}$-theory consisting of the axioms for abelian groups: AG $= \{$
  $\forall x, y, z.(x + y) + z = x + (y + z)$ (associativity)
  $\forall x, y.x + y = y + x$ (commutativity)
  $\forall x.x + 0 = x$ (neutral element)
  $\forall x.x + (-x) = 0$ (inverse element)
  $\}$.
  Thus: An $L_{\mathrm{ag}}$-structure is a model of AG iff it is an abelian group.
- Similarly, one defines the $L_{\mathrm{ring}}$-theory of rings, the $L_{\mathrm{ring}}$-theory of fields, $L_{\mathrm{ord}}$-theory of ordered sets, the $L_{\mathrm{oring}}$-theory of ordered rings etc.

# More examples of theories

Examples 2.3.

- The theory ACF of algebraically closed fields consists of:
  - the field axioms
  - "Every non-constant polynomial has a 0."
    Q: Can this be expressed by an $L_{\mathrm{ring}}$-sentence?
    A: Not by one, but by many: For each $n \geq 1$, take the axiom:
    $$\forall a_0 \ldots a_{n-1}. \exists x. x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0.$$
- For a prime $p$, the theory of algebraically closed fields of characteristic $p$ is:
  $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\underbrace{1 + 1 + \ldots + 1}_{p} = 0\}$
- The theory of algebraically closed fields of characteristic 0 is:
  $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{1 + 1 \neq 0, 1 + 1 + 1 \neq 0, 1 + 1 + 1 + 1 + 1 \neq 0, \text{ etc.}\}$
- What about axiomatising $\mathbb{R}$ (in $L_{\mathrm{oring}}$)?
  - $\mathbb{R}$ is the unique ordered field such that each bounded subset has a supremum.
  - Thus need to say: $\forall X \subseteq \mathbb{R}. X$ bounded $\rightarrow X$ has a supremum.
  - Not an $L_{\mathrm{oring}}$-formula: can only quantify over elements, not over subsets!

# Consistency, equivalence, implication

Definition 2.4. Fix a language $L$.

- An $L$-theory $T$ is consistent if (non-empty) models of $T$ exist.
- Two $L$-theories are equivalent if they have the same models.
- An $L$-sentence $\sigma$ follows from, or is implied by, an $L$-theory $T$ if $\sigma$ holds in every model of $T$. In that case, we write $T \vdash \sigma$.

Examples 2.5.

- All previous examples $(\mathrm{AG}, \mathrm{ACF}, \mathrm{ACF}_0, \mathrm{ACF}_p, \ldots)$ are consistent. The $L_{ag}$-theory $\{0 \neq 0\}$ is not inconsistent.
- Let $\mathrm{AG}'$ be AG with $\forall x.0 + x = x$ in place of $\forall x.x + 0 = x$. The models are in both cases all abelian groups, hence AG and $\mathrm{AG}'$ are equivalent.
- In every algebraically closed field, any monic polynomial of degree 2 has exactly two solutions. Thus: $\mathrm{ACF} \vdash \forall a_1 a_0.\exists^{=2} x.x^2 + a_1 x + a_0 = 0$.

Exercise 2.6. Prove that TFAE:

- $T$ is inconsistent.
- $T \vdash \bot$. (Recall: $\bot$ is the always-false sentence.)
- There exists a sentence $\sigma$ s.t. $T \vdash \sigma$ and $T \vdash \neg\sigma$.

# Complete theories

Definition 2.7. A consistent $L$-theory $T$ is complete if for every $L$-sentence $\sigma$, either $T \vdash \sigma$ or $T \vdash \neg\sigma$.

Example 2.8. Easiest way to construct a complete theory: consider the theory of a given $L$-structure $M$ $\mathrm{Th}(M) := \{\sigma \ L\text{-sentence} : M \models \sigma\}$ consisting all $L$-sentences true in the $L$-structure $M$.

Examples 2.9.

- $\mathrm{ACF}_0$ and $\mathrm{ACF}_p$ for every prime $p$ (in $L_{\mathrm{ring}}$), proof later.
- The theory DLO of dense linear orders without endpoints (in $L_{ord}$), proof later.
- The theory RCF of *real closed fields* (in $L_{\mathrm{oring}}$).
- The theory $\mathrm{VS}_K$ of infinite $K$-vector spaces (in $L_{K\text{-vect}}$).
- The theory RG of the *random graph* (in $L_{\mathrm{graph}}$).

- So far, we added more and more axioms to the theory of fields: algebraically closed fields (ACF), and then algebraically closed fields of fixed characteristic ($\mathrm{ACF}_0$, $\mathrm{ACF}_p$).
    - Example 2.9 tells us that it would have been impossible to go further: for any $L_{\mathrm{ring}}$-sentence $\sigma$, we have $\mathrm{ACF}_0 \vdash \sigma$ or $\mathrm{ACF}_0 \vdash \neg\sigma$. Thus, $\mathrm{ACF}_0 \cup \{\sigma\}$ is either equivalent to $\mathrm{ACF}_0$ or it is inconsistent.

# Elementary equivalence & substructures

- "A theory $T$ is complete" means: for any two models $M_1$ and $M_2$ of $T$, we have $\mathrm{Th}(M_1) = \mathrm{Th}(M_2)$. When this happens, we say:

Definition 2.10. Two $L$-structures $M_1$ and $M_2$ are elementarily equivalent if $\mathrm{Th}(M_1) = \mathrm{Th}(M_2)$. This is denoted by $M_1 \equiv_L M_2$ or just $M_1 \equiv M_2$.

Exercise 2.11. Prove that if two $L$-structures $M_1$ and $M_2$ are isomorphic then they are elementarily equivalent. The converse does NOT hold!

Examples 2.12.

- Any two algebraically closed fields of the same characteristic are elementary equivalent.
- Any two infinite $K$-vector spaces are elementary equivalent (so if $K$ is infinite, $K^2 \equiv K^3$!).

Definition 2.13. An $L$-embedding $e : M \to N$ between two $L$-structures $M$ and $N$ is an elementary embedding if it is an inclusion map such that

- $M \models \phi(a_1, \ldots, a_n) \Leftrightarrow N \models \phi(e(a_1), \ldots, e(a_n))$ for any $\phi(\bar{x})$ and $\bar{a} \in M^n$.

If $M$ is a substructure of $N$ and an $e : M \to N$ is an elementary embedding, then $M$ is called an elementary substructure of $N$. In this case we write $M \preceq N$.

# A strategy for completeness: Quantifier Elimination

- How can we prove that $\mathrm{ACF}_0$ is complete? We need to understand *everything* that can be expressed by an $L_{\mathrm{ring}}$-sentence. Hence, *all* definable sets.
- Philosophy: quantifiers are the source of all problems.
  But do we really need quantifiers?

Definition 2.14. Fix an $L$-theory $T$.

- Two formulas $\phi(\overline{x})$, $\psi(\overline{x})$ are equivalent modulo $T$ if for every $M \models T$, we have $\phi(M) = \psi(M)$ (equivalently, if $T \vdash \forall \overline{x}.\phi(\overline{x}) \leftrightarrow \psi(\overline{x})$).
- $T$ has quantifier elimination (q.e.) if every $L$-formula is equivalent modulo $T$ to a quantifier-free $L$-formula (i.e., a formula without $\forall$, $\exists$).

Exercise 2.15. For every $L$-structure $M$, we have:

- $\phi(\overline{x})$, $\psi(\overline{x})$ are equivalent modulo $\mathrm{Th}(M)$ iff $\phi(M) = \psi(M)$.
- $\mathrm{Th}(M)$ has q.e. $\Longleftrightarrow$ every $L$-definable set in $M$ can be defined by a quantifier-free formula.

# A strategy for quantifier elimination

Fix a theory $T$. How can one prove that $T$ has q.e.?

*Proposition 2.16.* *$T$ has q.e. $\iff$ every formula of the form $\exists y.\psi(\overline{x}, y)$, where $\psi$ is quantifier free, is equivalent modulo $T$ to a quantifier free formula.*

Proof.

- Let an arbitrary $L$-formula $\phi$ be given. Our goal is to get rid of quantifiers in $\phi$.
- Get rid of all "$\forall$" using that $\forall y.\phi(\overline{x}, y)$ is equivalent to $\neg\exists y.\neg\phi(\overline{x}, y)$.
- By induction on the length of $\phi$, we may replace all subformulas $\psi$ with equivalent quantifier free ones.
- We consider the various possibilities for $\phi$ from the definition of formulas:
  - if $\phi$ is atomic, it is quantifier-free.
  - if $\phi$ is $\psi_1 \vee \psi_2$, the sub-formulas $\psi_1, \psi_2$ are quantifier free, so $\phi$ is, too.
  - likewise for $\psi_1 \wedge \psi_2$, $\neg\psi_1$;
  - if $\phi(\overline{x})$ is $\exists y.\psi(\overline{x}, y)$, the subformula $\psi$ is quantifier free, so $\phi$ is equivalent to a quantifier free formula by the assumption. □

# Q.e. in ACF; completeness of $\mathrm{ACF}_0$, $\mathrm{ACF}_p$

Let $K$ be an algebraically closed field.

Definition 2.17. A subset of $D \subseteq K^n$ is constructible if it is a boolean combination of zero set of polynomials, i.e., if it definable with a quantifier free formula.

*Theorem 2.18 (Chevalley). The projection $K^{n+1} \to K^n$ of a constructible set is constructible.*

*Corollary 2.19.* ACF *has quantifier elimination.*

Proof. If $D \subseteq K^{n+1}$ is defined by $\phi(\overline{x}, y)$, then its projection to $K^n$ is defined by $\exists y \phi(\overline{x}, y)$. By Chevalley's theorem and Proposition 2.16, ACF has q.e. □

*Corollary 2.20.* $\mathrm{ACF}_0$ *and* $\mathrm{ACF}_p$ *are complete.*

Proof.

- Let $\sigma$ be any $L$-sentence; by q.e we may assume that $\sigma$ is quantifier free.
- Thus, $\sigma$ consists of $0, 1, +, -, \cdot, =, \top, \bot, \vee, \wedge, \neg$. (No variables!)
- In ACF, each term is equal to a sum $1 + \ldots + 1$.
- The characteristic of $K$ determines when such terms are equal or different; this decides whether $\sigma$ is true or false. □

# Vaught's test and the back-and-forth method

We next see another strategy to prove that a theory is complete.

Definition 2.21. Let $\kappa$ be an infinite cardinal and $T$ be a theory with models of size $\kappa$. $T$ is called $\kappa$-categorical if it has, up to isomorphism, a unique model of cardinality $\kappa$.

*Theorem 2.22 (Vaught's test). Let $T$ be a consistent $L$-theory with no finite models which is $\kappa$-categorical for some infinite cardinal $\kappa \geqslant |L|$. Then $T$ is complete.*

Exercise 2.23. Once you have seen the Löwenheim-Skolem Theorems (Lecture 4), use them to prove Vaught's test.

Showing $\kappa$-categoricity is often done using a technique called back-and-forth.

Definition 2.24. A partial isomorphism of two $L$-structures $M$ and $N$ is an isomorphism $f : M_0 \to N_0$ between two substructures $M_0 \subseteq_L M$, $N_0 \subseteq_L N$.

Definition 2.25. Let $\mathcal{I}$ be a set of partial isomorphisms of $M$ and $N$. We say that $\mathcal{I}$ has the back-and-forth property if for all $f \in \mathcal{I}$:

- forth: for all $m \in M$, there is $f \subseteq f' \in \mathcal{I}$ with $m \in \mathrm{dom}(f')$;
- back: for all $n \in N$, there is $f \subseteq f' \in \mathcal{I}$ with $n \in \mathrm{range}(f')$.

# Completeness of DLO

Definition 2.26. DLO is the $L_{\mathrm{ord}}$-theory of dense linear orders without endpoints.

- dense: for all $a < b$ there exists $c$ in between, i.e. $a < c < b$.
- linear: for all $a, b$ either $a < b$, $a = b$ or $b < a$.
- without endpoints: there are no minimal and no maximal elements.

Example 2.27. $\mathbb{Q} \models \mathrm{DLO}$, $\mathbb{R} \models \mathrm{DLO}$.

*Theorem 2.28. The theory* DLO *is $\aleph_0$-categorical and complete.*

Proof. Let $A, B \models \mathrm{DLO}$ be countable with one-to-one enumerations $a_0, a_1, \ldots$ and $b_0, b_1 \ldots$. We build a sequence $f_0 \subseteq f_1 \subseteq \ldots$ of partial isomorphisms $f_i : A_i \to B_i$ for $A_i \subset A$ and $B_i \subset B$ finite s.t. $f = \bigcup f_i$ is an isomorphism between $A = \bigcup A_i$ and $B = \bigcup B_i$.

As DLO has no finite models, completeness follows from Theorem 2.22. □

**Exercises 2.29.** Let $L$ be any language. Specify an $L$-theory $T$ such that the models of $T$ are exactly the infinite $L$-structures.

1. Let $T$ be a consistent $L$-theory. Prove that the following are equivalent:
   1. $T$ is complete;
   2. any two models of $T$ are elementarily equivalent;
   3. $T$ is equivalent to $\mathrm{Th}(M)$ for some model $M$ of $T$.
2. An ordered field $R$ is called real closed if every positive element of $R$ has a square root and every odd-degree polynomial has a zero. We denote the $L_{\mathrm{oring}}$-theory of real closed fields by RCF.
   - Axiomatise real closed fields with $L_{\mathrm{oring}}$-sentences.
   - By a result of Tarski, RCF has quantifier elimination in $L_{\mathrm{oring}}$. Therefore, (as for ACF) one deduces that RCF is complete in $L_{\mathrm{oring}}$. However, RCF does *not* eliminate quantifiers in $L_{\mathrm{ring}}$: what are the quantifier free $L_{\mathrm{ring}}$-definable sets?
3. Prove that the $L_{\mathrm{ag}}$-theory AG of abelian groups is not complete.
4. Write the field axiom "every non-zero element is invertible" in $L_{\mathrm{ring}}$.
5. Find quantifier free formulas equivalent to the ones below:
   1. $\phi(x, y) : \exists y. x \cdot y = 1$ in $\mathbb{C}$ seen as an $L_{\mathrm{ring}}$-structure;
   2. $\phi(x, y) : \exists x. y = x \cdot x$ in $\mathbb{R}$ seen as an $L_{\mathrm{oring}}$-structure;
   3. $\phi(a, b) : \exists x. x^2 + ax + b = 0$ in $\mathbb{R}$ seen as an $L_{\mathrm{oring}}$-structure.
6. Let $L = \{S\}$, where $S$ is a unary function, and let $\mathbb{Z}$ be an $L$-structure with $S$ interpreted as the function $a \mapsto a + 1$. Prove that the $L$-theory $\mathrm{Th}(\mathbb{Z})$ has q.e.

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