OUTLINE AND REFERENCES FOR MINI-COURSE ON HIGHER TOPOS THEORY (HTT–UF SUMMER SCHOOL: LEEDS, JUNE 2019)

CHARLES REZK

1. Outline

The basic structure of the lecture series is as follows.

Lecture 1: *What is a higher topos?*
We give a definition, motivated as the "∞-categorical generalization of topological space".

Lecture 2: *Homotopy theory in a higher topos.*
Truncation, connnectivity, homotopy sheaves, and more.

Lecture 3: *Descent in a higher topos.*
We describe various versions of "descent", leading to the "Giraud theorem" and the "object classifier".

Lecture 4: *Properties of a higher topos.*
A grab-bag of various useful concepts, including hypercompleteness and homotopy dimenison.

Lecture 5: *Maps of higher topoi and classifying topoi.*
Discussion of the notion of geometric morphism and computation of some interesting examples.

2. References

None of these references are "necessary". However, it would be good to come in (i) having some idea about what higher category theory is about, and (ii) knowing what a sheaf of sets on a topological space is.

2.1. **Higher category theory: surveys.** By "higher categories", I mainly mean what are usually called "∞-categories" or "$(\infty, 1)$-categories". There are a number of models of these, though the main one that people actually use are "quasicategories", whose theory was developed by André Joyal and Jacob Lurie. Note: when Lurie writes "∞-category", he means quasicategory, and this terminology has been taken up by many people, but not all people.

There is a lot of hard work needed to set up a model of higher category theory sufficient to do actual work. This has been done, e.g., the first 500 pages of Lurie's *Higher topos theory*, together with additonal hundreds of pages from *Higher algebra*. This is a lot of stuff, and it cannot be digested easily. In practice, it seems that people can often get a good "high-level" feeling for how to do ∞-categorical things, without wading deeply into the weeds: the slogan is it amounts to "ordinary categories + homotopy theory". However, to get such a feeling, you probably have to be fairly familiar with the essenials of homotopy theory.

Here are a few surveys which get you quickly to a high-level, though of course avoiding many details.

- Omar Antolín Camarena, "A whirlwind tour of the world of $(\infty, 1)$-categories", arXiv:1303.4669.
  This touches a lot of topics very quickly.

Date: May 24, 2019.

- Moritz Groth, "A short course on ∞-categories", arXiv:1007.2925.
  This surveys ∞-categories using the quasicategory model. Note in particular Chapter 3 on presentable ∞-categories; this topic is crucial for the definition of higher topoi.

2.2. Quasicategories.

- André Joyal, "Quasicategories and Kan complexes", Journal of Pure and Applied Algebra 175, 2002.
  The first paper introducing quasicategories as a model for (∞)-categories. It proves a key result: ∞-groupoids in this model are precisely Kan complexes, i.e., homotopy types.
- Denis-Charles Cisinksi, *Higher categories and homotopical algebra*, Cambridge University Press, 2019, Cisinski's website.
  This is the best available self-contained treatment of quasicategories *ab initio*.
- Jacob Lurie, *Higher topos theory*, Princeton University Press, 2009; also Lurie's website. Chapters 1–5 and appendices.
  The essential reference for quasicategories.

2.3. Higher category theory: other models and axiomatic approaches.

- Emily Riehl and Dominic Verity, "Elements of ∞-category theory", book-in-progress, Riehl's website.
  Describes a "model-independent approach" to working in ∞-categories, which includes quasicategories.
- Julia E. Bergner, "A survey of $(\infty, 1)$-categories", arXiv:math/0610239.
  Describes and relates various models for ∞-categories.

2.4. Classical topos and locale theory.

- Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer, 1994.
  This is my favorite introduction to topos theory; it is especially friendly to beginners.
- P.T. Johnstone, *Topos theory*, Academic Press, 1977 (reprinted by Dover).
  An older standard reference.
- P.T. Johnstone, "The point of pointless topology", Bulletin of the AMS 8 (1983).
  A brief introduction to locales.

2.5. Higher topos theory: surveys and references.

- Bertrand Töen and Gabriele Vezzosi, "Homotopical Algebraic Geometry I: Topos theory", Advances in Math, 2004; arXiv:math/0207028.
  The first paper to introduce higher topoi. It uses similicially enriched categories as its foundations for higher categories. It also discusses model topoi. Note: Töen and Vezzosi assume hypercompleteness as an axiom for higher topoi.
- Jacob Lurie, "On ∞-topoi", arXiv:math/0306109.
  The account of higher topos theory from Lurie's thesis. This is the core of what became Chapters 6–7 of Lurie's HTT book, but is agnostic with respect to higher categorical foundations. Very readable.
- Jacob Lurie, *Higher topos theory*, Princeton University Press, 2009; also Lurie's website. Chapters 6–7.
  The essential reference for higher topoi.
- Charles Rezk, "Toposes and homotopy toposes", Rezk's website.
  Notes on model topoi.

2.6. Other stuff about higher topoi that seems interesting.

- Thomas Nikolaus, Urs Schreiber, Danny Stevenson, "Principal infinity-bundles – general theory", arXiv:1207.0248.

  A nice treatment of principal bundles using the general notion of descent in an ∞-topos.
- Mathieu Anel, Georg Biedermann, Eric Finster, André Joyal, "A generalized Blakers-Massey theorem", arXiv:1703.09050.

  This gives a nice review of the notion of "descent" in an ∞-topos, as well as some interesting applications; e.g., the Blakers-Massey theorem.
- Mathieu Anel, Damien Lejay, "Exponentiable Higher Toposes", arXiv:1802.10425.

  What it says in the title.
- Jacob Lurie, *Spectral algebraic geometry*, book-in-progress, Lurie's website. Appendix A.

  The notion of a coherent ∞-topos.

Department of Mathematics, University of Illinois, Urbana, IL
Email address: rezk@illinois.edu