Commuting planar polynomial derivations for conservative Newton systems

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joint work with Joel Nagloo and Alexey Ovchinnikov

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Introduction

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Let d and δ be S-derivations on $R \supset S$. We say that d and δ **commute** if

- $\forall a \in R \quad d(\delta(a)) = \delta(d(a))$

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- A K-derivation d on $K[x, y]$ is uniquely determined by $d(x)$ and $d(y)$.
- K-derivations d and δ on $K[x, y]$ commute if and only if $d(\delta(x)) = \delta(d(x))$ and $d(\delta(y)) = \delta(d(y))$.

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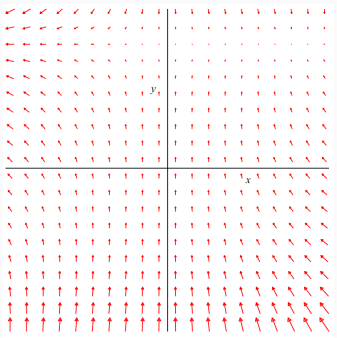

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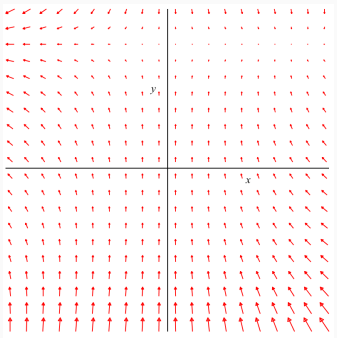

Two $\mathbb{R}$-derivations d and δ commute if and only if the flows of the corresponding vector fields commute.

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In this case, H is a first integral of d.

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Answer: If $\deg f \leqslant 1$, yes. If $\deg f > 1$, no.

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Let d be the K-derivation $d(x) = y$, $d(y) = f(x)$

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The main result can't be extended to all Hamiltonian derivations! E.g.

$$\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + x^2 \\ -2xy \end{pmatrix} \text{ and } \delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \text{ commute.}$$

γ is Hamiltonian with $H_\gamma = y(1 + x^2)$.

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2. The Jacobian conjecture for $\mathbb{C}$ on 2 variables has an equivalent statement in terms of $\mathbb{C}$-derivations on $\mathbb{C}[x, y]$.

Motivation: Solving ODE

Theorem

Consider the initial value problem

$$\dot{x} = f_1(x, y),\ \dot{y} = f_2(x, y),\ x(0) = x_0,\ y(0) = y_0.$$

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commute and are transversal at (x_0, y_0), then the initial value problem has a solution given by

$$(x(t), y(t)) = F^{-1}(t, 0),$$

where

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \int_{x_0}^{x} \frac{g_2(r,y)}{\Delta(r,y)}\,dr + \int_{y_0}^{y} \frac{-g_1(x_0,s)}{\Delta(x_0,s)}\,ds \\ \int_{x_0}^{x} \frac{-f_2(r,y)}{\Delta(r,y)}\,dr + \int_{y_0}^{y} \frac{f_1(x_0,s)}{\Delta(x_0,s)}\,ds \end{pmatrix},\ \text{and } \Delta(x, y) = \begin{vmatrix} f_1(x, y) & g_1(x, y) \\ f_2(x, y) & g_2(x, y) \end{vmatrix}.$$

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If $\mathbb{C}$-alg. hom. $F : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$, $x \mapsto F_1, y \mapsto F_2$ satisfies $det(J(F_1, F_2)) \in \mathbb{C}^*$,

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Corollary to Main Result

$d(x) = y, d(y) = f(x)$ is not part of a commutative basis.

Related Results

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$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x + q_0(x) + yq_1(x) + \ldots + y^n q_n(x) \end{pmatrix}, \; ord(q_0) \geqslant 2.$$

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Is there a transversal commuting δ?

If $n \geqslant 4$: No. (Volokitin & Ivanov)

If $n = 3$: Yes iff $d(y) = (-x - a^2x^3)(1 + hy) - 3axy(1 + hy)^2 - h'y^3$,

$$a \in K, h \in K[x] \text{ (V. \& I.)}$$

If $n = 2$:

Related Results: Newton Systems with SHO Linearization

$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x,y) \end{pmatrix}$ corresponds to the differential equation $\ddot{x} = f(x, \dot{x})$.

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If $n = 2$: No. (V. & I.)

If $n = 1$: unknown?

If $n = 0$: Yes iff $d(y) = -x$. (Amel'kin)

Proof

Proof

Fix $f \in K[x]_{\geqslant 2}$. Let $\delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f \end{pmatrix}$.

Suppose γ commutes with δ and write $\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{M} c_i y^i \\ \sum_{i=0}^{M} d_i y^i \end{pmatrix}$, where the c_i and d_i are unknown elements of $K[x]$.

Write $\delta(\gamma(x)) = \gamma(\delta(x))$ and $\delta(\gamma(y)) = \gamma(\delta(y))$. Expanding these equations yields

$$\begin{pmatrix} c'_M y^{M+1} + c'_{M-1} y^M + \ldots + c'_0 y \\ d'_M y^{M+1} + d'_{M-1} y^M + \ldots + d'_0 y \end{pmatrix} + \begin{pmatrix} M f c_M y^{M-1} + \ldots + f c_1 \\ M f d_M y^{M-1} + \ldots + f d_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ f' & 0 \end{pmatrix} \begin{pmatrix} c_M y^M + \ldots + c_0 \\ d_M y^M + \ldots + d_0 \end{pmatrix}.$$

Proof

Equating coefficients of like powers of y in each component yields the following system of first-order linear ODEs in c_i, d_i (actually two independent systems):

$$
\begin{aligned}
c'_M &= 0 \\
d'_{M-1} &= f' c_M \\
c'_{M-2} &= -Mfc_M + d_{M-1} \\
d'_{M-3} &= -(M-1)fd_{M-1} + f' c_{M-2} \\
&\vdots
\end{aligned}
\qquad
\begin{aligned}
d'_M &= 0 \\
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c'_{M-3} &= -(M-1)fc_{M-1} + d_{M-2} \\
&\vdots
\end{aligned}
$$

This means that γ can be written as $\gamma = \gamma_1 + \gamma_2$, where γ_1 and γ_2 each commute with δ.

We will show that $\gamma_1 \in K[H] \cdot \delta$ and $\gamma_2 = 0 \cdot \gamma$.

Proof

If M is odd, the systems are

$$
\begin{aligned}
c'_M &= 0 \\
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c'_{M-2} &= -Mfc_M + d_{M-1} \\
d'_{M-3} &= -(M-1)fd_{M-1} + f' c_{M-2} \\
&\vdots \\
d'_0 &= -2fd_2 + f' c_1 \\
fc_1 &= d_0
\end{aligned}
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c'_{M-3} &= -(M-1)fc_{M-1} + d_{M-2} \\
&\vdots \\
c'_0 &= -2fc_2 + d_1 \\
fd_1 &= f' c_0
\end{aligned}
$$

If M is even, the systems are

$$
\begin{aligned}
c'_M &= 0 \\
d'_{M-1} &= f' c_M \\
c'_{M-2} &= -Mfc_M + d_{M-1} \\
d'_{M-3} &= -(M-1)fd_{M-1} + f' c_{M-2} \\
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$$

Proof

We focus on the upper-right system:

$$
\begin{aligned}
d'_M &= 0 \\
c'_{M-1} &= d_M \\
d'_{M-2} &= -Mfd_M + f'c_{M-1} \\
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&\vdots \\
c'_0 &= -2fc_2 + d_1 \\
fd_1 &= f'c_0
\end{aligned}
$$

We have $M+1$ variables: $d_M, c_{M-1}, d_{M-2}, \ldots, d_1, c_0$ and $M+2$ equations.

Observe that the first $M+1$ equations form a "triangular" system, which can be solved as follows:

$d_M = \text{const.}$, $c_{M-1} = \int d_M dx + \text{const.}$,
$d_{M-2} = \int(-Mfd_M + f'c_{M-1})dx + \text{const.}$, etc.

Proof

We show that any such solution cannot satisfy the equation $fd_1 = f'c_0$.

Focus on the leading terms of d_M, etc. Let $N = \deg f$.

var.	"LC"
d_M	d_M
c_{M-1}	d_M
d_{M-2}	$d_M \cdot LC(f) \cdot (-M+N)/(N+1)$
c_{M-3}	$d_M \cdot LC(f) \cdot (-(M-1)+(-M+N)/(N+1))/(N+2)$
etc.	etc.

Setting the leading coefficient of fd_1 equal to that of $f'c_0$ shows that N must belong to the set

$$\{1\} \cup \left\{ -\tfrac{2k+1}{2k-1} \mid k \in \mathbb{Z}, 1 \leqslant k \leqslant \tfrac{M-1}{2} \right\} \cup \{-1\} \cup \left\{ -\tfrac{k}{k-1} \mid k \in \mathbb{Z}, k \geqslant 2 \right\}.$$

$\rightarrow\leftarrow$

Proof

Proving this for arbitrary M is non-trivial. We use the following fact:

Lemma

For every positive integer k, the K-derivation

$$\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x^{-\frac{2k+1}{2k-1}} \end{pmatrix}$$

of the ring $K[x^{-\frac{1}{2k-1}}, x^{\frac{1}{2k-1}}, y]$ commutes with the K-derivation

$$\beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum_{l=0}^{k} \mathbf{a}_{2(k-l)} x^{1+(1-\frac{2k+1}{2k-1})l} y^{2(k-l)} \\ \sum_{l=0}^{k} \mathbf{a}_{2(k-l)+1} x^{(1-\frac{2k+1}{2k-1})l} y^{2(k-l)+1} \end{pmatrix},$$

where the $\mathbf{a}_i \in K$ are defined recursively as follows: $\mathbf{a}_{2k+1} \in K \setminus \{0\}$ is arbitrary, $\mathbf{a}_{2k} = \mathbf{a}_{2k+1}$, and for $0 < l \leqslant k$,

$$\mathbf{a}_{2(k-l)+1} = \left(-\tfrac{2k+1}{2k-1}\mathbf{a}_{2(k-l)+2} - (2(k-l)+3)\mathbf{a}_{2(k-l)+3}\right)\left((1-\tfrac{2k+1}{2k-1})l\right)^{-1}$$

$$\mathbf{a}_{2(k-l)} = \left(\mathbf{a}_{2(k-l)+1} - (2(k-l)+2)\mathbf{a}_{2(k-l)+2}\right)\left((1-\tfrac{2k+1}{2k-1})l+1\right)^{-1}.$$

Future Work

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Problem 1: (with Jason Bell)

What is the centralizer of $y\partial_x + f(x)\partial_y$ in the second Weyl algebra? ($\mathbb{C}[x, y, \partial_x, \partial_y]$)

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1. Applications of higher-order symmetries to solving differential equations.

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Problem 2:

Find a useful way of describing the K-derivations that are equivalent to $y\partial_x + f(x)\partial_y$.