Talk abstracts

A family of integrable hierarchies containing the discrete KdV and the Intermediate Long Wave hierarchies.

Alexandr Buryak, University of Leeds

I will talk about a new one-parameter family of integrable hierarchies with two dependent variables, containing, as reduction, the discrete KdV hierarchy and the hierarchy of the Intermediate Long Wave equation. This hierarchy controls certain topological invariants of the moduli space of algebraic curves with spin structure and is constructed using the theory of hierarchies of topological type, developed by Dubrovin and Zhang.

PreHamiltonian and Hamiltonian rational difference operators

Sylvain Carpentier, Columbia University

Following J.P. Wang's talk, we will apply our results on preHamiltonian pairs of difference operators to the differential-difference equation recently discovered by Adler and Postnikov. In particular, we will see how the corresponding Nijenhuis recursion operator generates an infinite hierarchy of symmetries. We will illustrate our theory with more examples if time permits. Finally, we will investigate the relations between preHamiltonian pairs and Hamiltonian operators, both local and rational. In particular, we will explain how an equation for which the recursion operator is given by a preHamiltonian pair and which is hamiltonian must be bi-hamiltonian.

Hypergeometric equation, Darboux factorisations, and bispectrality

Oleg Chalykh, University of Leeds

We will describe differential-difference bispectral families that can be constructed starting from the hypergeometric equation. This can be viewed as a generalisation of Wilson's adelic Grassmannian and its trigonometric analogue studied by Haine, Horozov and Iliev. The main problem is to describe all possible rational factorisations (or higher order Darboux transformations) that can be constructed starting from the hypergeometric equation. This is achieved by analysing the monodromy of the (powers of the) hypergeometric equation by representation-theoretic methods. This is joint work with A Khalid (Leeds).

Classical affine W-algebras for classical Lie algebras and associated integrable systems

Alberto De Sole, Sapienza University of Rome

I will describe the construction of a Lax type operator L(d) with coefficients in the classical affine W-algebra W(g,f), satisfying an Adler type identity. This operator is used to construct an integrable Hamiltonian hierarchy of Lax type equations. All the result presented are joint work with V. Kac and D. Valeri.

How to test that a special function is differentially transcendent?

Thomas Dreyfus, CNRS, IRMA, Strasbourg

Consider a solution of an order two equation of the form

$$\sigma^2 y + a\sigma y + by = 0,$$

where, a,b belong to a convenient field k, and σ is an automorphism of k. Typical examples are

- $k = \mathbb{C}(x)$, $\sigma : f(x) \mapsto f(x+1)$;
- $k = \mathbb{C}(x)$, $\sigma : f(x) \mapsto f(qx)$.

To such equations, we may associate a group that measures the algebraic and differential relations among the solutions. Our goal is to give a criterion that is as minimal as possible to ensure that any solution of the equation satisfies no algebraic differential equation. We will apply our criterion to the elliptic analogue of the hypergeometric equation.

Integrable derivations of hypersurfaces in characteristic p

Eleonore Faber , University of Leeds

Let k be a commutative ring and A a commutative k-algebra. A k-linear derivation δ of A is called n-integrable, where n is a positive integer or $n = \infty$, if it extends up to a Hasse–Schmidt derivation of A over k of length n.

In this talk let k be a field of characteristic $p > 0$. While over a field of characteristic 0 any derivation is integrable, this question is much more delicate over positive characteristic fields. We study the module of n-integrable derivations along $X = \text{Spec}(A)$, for some classes of quasi-homogeneous hypersurfaces X. We define the notion of hops of modules of n-integrable derivations. The number of hops seems to be an interesting new invariant of A, intrinsic to characteristic p.

This is joint work with Angelica Benito.

From double brackets to integrable systems

Maxime Fairon, University of Leeds

Double brackets were introduced by Van den Bergh in his successful attempt to understand the Poisson geometry of (multiplicative) quiver varieties directly at the level of the path algebra of a quiver. I will review the basics of this theory and illustrate how it works in the simple case of an extended Jordan quiver. In particular, I will explain how it helps to easily derive integrable systems in the Calogero-Moser family.

Patching in Differential Galois Theory

Julia Hartmann, University of Pennsylvania

Abstract

Patching methods have been very fruitful in Galois theory for several decades. More recently, they have found use in differential Galois theory. The talk will give a brief introduction to patching and then focus on a survey of those applications. This includes solutions to certain inverse problems and embedding problems. The latter relate to the question of freeness of the absolute differential Galois group (see Michael Wibmer's talk).

D-orthogonal polynomials, integrable systems and matrix models

Emil Horozov,
Sofia University & Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences

Abstract

The bispectral operators have a number of specific properties that connect them to other areas of research. Simple but very important examples of bispectral operators are the Jacobi, Laguerre and Hermite differential operators. The corresponding polynomial systems are their eigenfunctions, which also satisfy a 3-term recurrence relation of the form

$$xP_n(x) = P_{n+1} + \gamma_0(n)P_n(x) + \gamma_1(n)P_{n-1}.$$

They are important tools in mathematics, physics, engineering, etc. and are called classical orthogonal polynomials. In the last decades it was revealed that they play fundamental role in matrix models.

These polynomials were recently generalized [1,2] to include polynomial systems $P_n(x)$ satisfying recurrence relations of higher order:

$$xP_n(x) = U(n)P_n \stackrel{def}{=} P_{n+1}(x) + \sum_{j=0}^{d} \gamma_j(n)P_{n-j}(x).$$

Polynomial systems with type are known as d-orthogonal. The property is equivalent to orthogonality with respect to several (d) measures, which gives their name.

If $P_n(x)$ are eigenfunctions of a differential operator

$$L = \partial_x^{d+1} + \sum_{j=0}^{d} a_j(x)\partial_x^j$$

we again have bispectral operators L and U:

$$LP_n(x) = nP_n(x), \;\; UP_n(x) = xP_n(x).$$

The construction of the polynomial systems will be recalled for some simple cases, relevant for the talk. Our main example will be

$$L = -\partial_x^{d+1} + x\partial_x$$

As with many bispectral operators it turns out that some of the constructed here have a lot of interesting properties. Some of them are related again to matrix models.

The operator U can be represented as an semi-infinite matrix. It naturally defines a solution to the (bi-graded) Toda lattice hierarchy.

We will show that the solution naturally extends to a solution of the bi-infinite Toda hierarchy. The corresponding tau-functions are partition functions of matrix models, which are generalizations of Kontsevich-Penner model:

$$Z_n(\Lambda) = \mathcal{C}^{-1}\int_{\mathcal{H}_M} [d\Phi] \exp\Big[- Tr\big(\frac{\Phi^3}{3} - \frac{\Lambda^2\Phi}{2} + (n+1)\log\Phi\big)\Big],$$

where $\mathcal{H}_M$ is the space of $M \times M$ Hermitian matrices, $\Phi \in \mathcal{H}_M$, Λ is a diagonal matrix. For $n = -1$ this is the partition function of Kontsevich-Witten model in 2d-gravity.

Some of the partition functions are generating functions of topological invariants of the moduli space of Riemann surfaces, e.g. the open and closed intersection numbers, cf. [3,4].

References

[1] E. Horozov, *Vector orthogonal polynomials with Bochner's property*, Constr Approx (2017). `https://doi.org/10.1007/s00365-017-9410-6`, arXiv:1609.06151v1

[2] E. Horozov, d-Orthogonal Analogs of Classical Orthogonal Polynomials, SIGMA 14 (2018), 063, 27

[3] A. Alexandrov, Open intersection numbers, Kontsevich-Penner model and cut-and-join operators, J. High Energ. Phys. (2015) 2015: 28. `https://doi.org/10.1007/JHEP08(2015)028`

[4] A. Buryak. Open intersection numbers and the wave function of the KdV hierarchy. Moscow Mathematical Journal 16 (2016), no. 1, 27-44. arXiv:1409.7957.

Disintegrated Differential Equations

Rémi Jaoui, University of Waterloo

Disintegration is a property that arises naturally in the model-theoretic study of differential algebraic equations and that reflects strong algebraic-independence properties for its solutions. Intuitively, an algebraic differential equation is disintegrated when any algebraic relation involving any number of solutions (of the differential equation under study) can be reduced to an algebraic relation involving only two of them.

It is believed that, in any reasonable sense, a "typical" differential equation should be disintegrated. However, this property has paradoxically only been established for very specific algebraic differential equations. In my talk, I will present some effective tools that I have developed in my PhD and slightly after in order to study disintegration. I will then explain some applications to certain differential equations describing a geodesic motion on an Riemannian manifold with negative curvature.

Signatures of algebraic curves

Irina Kogan, North Carolina State University

We consider the group-equivalence problem for planar curves. This problem consists of deciding whether two given curves are related by a transformation from a certain group. Signature curves based on differential invariants have been proposed to solve the group-equivalence problem. In the case of algebraic curves, a signature curve can be described as the zero set of a defining polynomial in two variables, which we call the signature polynomial. However, explicit computation of the signature polynomial is often beyond a reasonable computational capacity. After reviewing the signature construction, I will show that the degree of the signature polynomial can be predicted without its explicit computation. I will also present some interesting examples and applications of the degree formula. Although many of the results are general, I will concentrate on the actions of classical geometric groups: projective, affine and Euclidean. These groups play an important role in computer vision and shape analysis.

This is a joint work with Michael Ruddy and Cynthia Vinzant from North Carolina State University.

Applications of differential invariants to solve differential equations

Wei Li, Dalian Ocean University

Differential invariants are the fundamental building blocks for constructing invariant differential equations and invariant variational problems, as well as determining their explicit solutions and conservation laws. In this talk, the computational algorithms, based on the method of equivariant moving frames of Fels and Olver, are introduced to obtain the complete classification of the differential invariants and their syzygies of the symmetry group of the nonlinear partial differential equations. The algorithms are efficient, requiring only linear algebra and differentiation, but no explicit formulas for either the moving frames, or the differential invariants and invariant differential operators, or even the Maurer-Cartan forms.

In the second half of the talk, based on the method of equivariant moving frames, I will show how to use the differential invariants and their syzygies to solve the ordinary differential equations and even the nonlinear partial differential equations. How to use the differential invariants to find the potential relations between the nonlinear partial differential equations and the linear differential equations or the classic solvable nonlinear differential equations, will be discussed too.

Transseries, Hardy fields, and surreal numbers

Vincenzo Mantova, University of Leeds

Transseries, such as LE series, arise when dealing with certain asymptotic expansions of real analytic function around essential singularities. A hard, but fundamental result by Aschenbrenner, van den Dries and van der Hoeven is that the differential field of LE series is *model complete*, and in particular, it contains formal solutions to a wide class of differential equations.

I will review the recent results by Berarducci and myself, and by Aschenbrenner, van den Dries, and van der Hoeven, linking transseries to Conway's *surreal numbers*: it turns out that surreal numbers form a large (a proper class!) field of transseries with a canonical derivation, which happens to be nonstandard model of the differential field of LE series containing isomorphic copies of every Hardy field, and they also form a natural domain where a large class of transseries "converge", giving them an immediate functional interpretation.

This is joint work with A. Berarducci.

Poisson structure on the space of polygons

Ian Marshall, Higher School of Economics, Moscow

A family of Poisson structures, parametrised by an arbitrary odd function φ, is defined on the space $\mathcal{W}$ of twisted polygons in $\mathbb{R}^{\nu}$ and Poisson reductions with respect to natural Poisson group actions on $\mathcal{W}$ are described. The lattice Virasoro structure, the second Toda lattice structure and some extended Toda lattice structures can be found to arise in this way. A general result is proved showing that, for any ν, to certain concrete choices of φ, there correspond compatible Poisson structures which generate all the extended bigraded Toda hierarchies of a suitable size.

Difference Galois theory over Non-Noetherian rings

Andreas Maurischat, RWTH, Aachen University

In difference Galois theory, the base difference ring is usually assumed to be a field or a finite product of fields. In this talk, we explain how the whole theory of Picard-Vessiot rings and Galois groups extends to any difference-simple ring. This includes non-Noetherian rings like the ring of periodic sequences of arbitrary period with the shift operator. In our setting, we also allow the endomorphism not to be bijective.

Algorithmic aspects of polynomial composition

Alice Medvedev, City University of New York

We know everything about adding and multiplying polynomials and rational functions; but throw in composition, and we know next to nothing.

A key ingredient of [1] is a partial group action of the symmetric group on the set of decompositions of a polynomial, initially defined for the transposition generators of the group and then extended to the rest of the group when possible.

For example, the polynomial $P(x) = x^{98}(x^{98}+1)^2$ has decompositions $A := (x(x^7+1)^2, x^7, x^2)$ and $B := (x^2, x(x^{14}+1), x^7)$; the second decomposition B comes from the first one via the action of the permutation $(123) = (12)(23)$, but the action of (12) on A is not defined because nothing like x^7 is a left compositional factor of $x^7(x^{49}+1)^2$.

I'll describe what we (re?)invented in this setting, in the hopes that some more computation-savvy audience member will recognize this as part of a well-known general theory.

[1] Invariant varieties for polynomial dynamical systems, with Thomas Scanlon, Ann. of Math., 179(1):81-177, 2014.

Differential transcendence of solutions of difference Riccati equations

Seiji Nishioka, Yamagata University

Historically, there is the paper by H. Tietze published in 1905, in which he studied differential transcendence of solutions of difference Riccati equations, $y(x+1) = (A(x)y(x)+B(x))/(C(x)y(x)+D(x))$. He proposed his normal form $y(x+1) = r(x)/y(x) + 1$ and studied the differential transcendence of solutions when $r(x)$ is a rational function with $r(x) \to 0$ ($x \to \infty$). His theorem says that if there is no rational function solution, there is no solution satisfying an algebraic differential equation. In this talk, I am going to introduce the essence of Tietze's treatment by using the discrete valuation, and an application to the q-Airy equation.

Integrability in 3D

Vladimir Novikov, Loughborough University

We consider the problem of detecting and classifying integrable partial differential (and difference) equations in 3D. Our approach is based on the observation that dispersionless limits of integrable systems in 3D possess infinitely many multi-phase solutions coming from the so-called hydrodynamic reductions. We consider a novel perturbative approach to the classification problem of dispersive equations. Based on the method of hydrodynamic reductions, we first classify integrable quasilinear systems which may (potentially) occur as dispersionless limits of soliton equations in 3D. To reconstruct dispersive deformations, we require that all hydrodynamic reductions of the dispersionless limit are inherited by the corresponding dispersive counterpart. This procedure leads to a complete list of integrable third and fifth order equations, which generalize the examples of Kadomtsev-Petviashvili, Veselov-Novikov and Harry Dym equations as well as integrable Davey-Stewartson type equations, some of which are apparently new.

We also consider the problem of dispersive deformations on the Lax representation level and thus show that our approach allows starting from the dispersionless Lax representations to construct the fully dispersive Lax pairs representing the fully dispersive integrable systems.

We extend this approach to the fully discrete case. Based on the method of deformations of hydrodynamic reductions, we classify discrete 3D integrable Hirota-type equations within various particularly interesting subclasses as well as a number of classification results of scalar differential-difference integrable equations including that of the intermediate long wave and Toda type.

Elimination of unknowns in systems of differential-algebraic equations

Alexey Ovchinnikov, City University of New York

We will discuss new results in elimination of unknonws in systems of differential-algberaic equations. Among other advantages, our new approach allows for randomized computation to significantly increase the efficiency and tackle problems that could not be tackled before. This approach is based on uniform upper bounds for numbers of differentiations. We will also discuss cases in which our bound is asymptotically tight.

This is joint work with Gleb Pogudin and Thieu Vo.

Elimination of unknowns in systems of difference equations

Gleb Pogudin, New York University

Systems of difference equations are actively used in modeling (for example, for discrete-time processes), combinatorics, and number theory. Elimination of unknowns is a fundamental tool for studying solutions of equations (linear, polynomial, differential, etc.). The elimination problem for difference equations can be stated as follows. Given a system of difference equations in $m + n$ unknowns, determine whether there is a consequence of the system involving only the first m variables and find such a consequence if it exists.

We solve this problem by deriving an explicit upper bound for the number of shifts sufficient for a reduction of this elimination problem to a well-studied elimination problem for systems of polynomial equations. As a special case, we derive a bound for determining the consistency of a system of difference equations (effective difference Nullstellensatz). Several similar bounds have been recently discovered for differential equations, but, for difference equations, our bound is the only such bound as far as we are aware.

This is a joint work with Alexey Ovchinnikov and Thomas Scanlon.

Role of the Weyl Algebra in Complete Integrability

Emma Previato, Boston University

Abstract

There remain major open problems stemming from the interaction between differential algebra and integrability originated in the 1960s and '70s with the "KP hierarchy". The talk will cover higher-rank commutative algebras of differential operators and their KP deformations. We will review old work (with G. Wilson and G. Latham) on "effectivization" of the theory, and current work by I. Burban and A. Zheglov that completes the classification (P.-Wilson) of rank-2 elliptic algebras. Revisiting tools of differential algebras (such as the differential resultant and Darboux transformation) aided by computation, we propose their use in studying open problems in the theory of Weyl algebras.

Cohomological field theories and integrable systems

Paolo Rossi, University of Burgundy

Cohomological field theories are families of cohomology classes on the moduli space of stable curves. They have to be compatible with the boundary strata structure of such spaces (smaller moduli spaces appear in the boundary of bigger ones). They are a very effective tool to investigate the topology of the moduli spaces, but they have also proved to be at the origin of the surprising emergence of integrable systems of PDEs which was first noticed in this context by Witten. Such integrable systems (in both the Dubrovin-Zhang and in the Double Ramification approaches) turn out to be Hamiltonian dispersive PDEs. However, in a joint work with Buryak, we noticed that the axioms of CohFT can be loosened to accommodate for more general, non-Hamiltonian, integrable systems. Using geometric arguments we found recursive formulas for the symmetries of such systems and several interesting examples.

Effective bounds in the realm of differential polynomials

Omar Leon Sanchez, University of Manchester

I will survey some recent results related to effective bounds in differential algebra. More concretely, I will present a bound on the "order" of a characteristic set of a prime differential ideal P that only depends on the "order" of the equations defining P. This bound yields a bound for the coefficients of the Kolchin polynomial of P, a question studied (but not fully answered) by Kolchin himself. Other applications of this bound is on uniformly bounding the size a finite solution set of a system of PDEs, this in turn can be used to prove that Zariski-closure and Kolchin-polynomials are definable in families of differential varieties (the latter uses a recent bound on the Hilbert-Kolchin index).

These results are a collection of joint work with J. Freitag, R. Gustavson, and W. Li.

Construction of integrable systems by means of the technique of separation of variables

Oleg Sheinman, Steklov Mathematical Institute, Moscow

A plane algebraic curve whose Newton polygone contains d integer points is completely determined by giving d points of the plane, the curve is passing through. Then its coefficients, regarded as functions of sets of coordinates of the points, are Poisson commuting with respect to any pair of coordinates corresponding to the same point. This has been observed by Babelon and Talon (2002). They reduced the statement to a special version of separation of variables. A result, more general in some respects, and less general in the others, is obtained by Enriquez and Rubtsov (2003). It follows as a particular case that the coefficients of the interpolation polynomial are Poisson commuting with respect to interpolation data. We prove a general statement in frame of separation of variables explaining all these facts. It is as follows: any (non-degenerate) system of n smooth functions of n+2 variables yields an integrable system with n degrees of freedom. Apart from already mentioned, the examples include a version of Hermit interpolation polynomial, systems related to Weierstrass models of curves (= miniversal deformations of singularities). If the time admits, I'll explain how a Hitchin system on a hyperelliptic curve can be given by means of this technique, and what is the relation of the last to recent works by Buchstaber and Mikhailov on the universal bundle of symmenric powers of hyperelliptic curves.

Liouvillian solutions of nonlinear differential equations

Varadharaj Ravi Srinivasan, IISER Mohali, Punjab

I will explain a structure theorem for Liouvillian extensions that is analogous to generalized elementary extensions (Singer 1975) and apply this result to study Liouvillian solutions of nonlinear differential equations.

Commuting planar polynomial vector fields for conservative Newton systems

Peter Thompson, City University of New York

We study the problem of characterizing polynomial vector fields that commute with a given polynomial vector field in the plane. It is a classical result that one can write down solution formulas for an ODE that corresponds to a planar vector field that possesses a linearly independent commuting vector field. Let f be an element of $K[x]$, where K is a field of characteristic zero, and let d be the K-derivation on $K[x, y]$ that corresponds to the differential equation $x'' = f$ in a standard way. This is called a conservative Newton system, as it describes the position of a particle under the influence of a conservative force. Let also H be the Hamiltonian polynomial for d, that is $H = y^2 - 2int(f)$, where $int(f)$ denotes the antiderivative of f with respect to x with zero constant term. It is known that the set of all polynomial K-derivations that commute with d forms a $K[H]$-module M_d. We show that, for every such d, the module M_d is of rank 1 if and only if the degree of f is at least 2.

PreHamiltonian and Nijenhuis rational difference operators

Jing Ping Wang, University of Kent

In this talk we introduce preHamiltonian pairs of difference operators and study their connections with Nijenhuis operators and the existence of weakly non-local inverse recursion operators for differential-difference equations. We begin with a rigorous setup of the problem in terms of the skew field of rational (pseudo-difference) operators over a difference field. In particular, we give a criteria for a rational operator to be weakly non-local. A difference operator is called preHamiltonian, if its image is a Lie subalgebra. Two preHamiltonian operators form a preHamiltonian pair if any linear combination of them is preHamiltonian. Then we show that a preHamiltonian pair naturally leads to a Nijenhuis operator, and a Nijenhuis operator can be represented in terms of a preHamiltonian pair.

Free differential Galois groups

Michael Wibmer, University of Pennsylvania

We will discuss recent progress towards understanding the absolute differential Galois group of the field of rational function. In particular, we will introduce free proalgebraic groups and explain their characterization in terms of embedding problems. This is joint work with Annette Bachmayr, David Harbater and Julia Hartmann.

Poster abstracts

The Quantum Auxiliary Linear Problem and Quantum Darboux-Bäcklund Transformations

Iain Findlay, Heriot Watt University

The aim of this work was to determine an algebraic method for finding the Lax pairs associated to the integrable systems found from the hierarchy of conserved quantities generated by the expansion of the transfer matrix. This was done for the cases of both closed and open boundary conditions, and the q-oscillator model was presented as an example. The Idea of quantum Darboux-Bäcklund transformations (in the context of continuous time) was also introduced.

Integration in Finite Terms with Dilogarithmic, Logarithmic Integrals and Error Functions

Yashpreet Kaur, IISER Mohali, Punjab

In this poster, we discuss an extension of Liouville's Theorem that includes dilogarithmic integrals, logarithmic integrals and error functions in its field of definition. Moreover, we provide a new proof for Baddoura?s theorem which neither assumes that F is a liouvillian extension of CF nor that CF is an algebraically closed field.

Full-parameter discrete Painlevé systems from non-translational Cremona isometries

Alexander Stokes, University College London

Since the classification of discrete Painlevé equations in terms of rational surfaces, there has been much interest in the range of integrable equations arising from each of the 22 surface types in Sakai's list. For all but the most degenerate type in the list, the surfaces come in families which admit affine Weyl groups of symmetries. Translation elements of this symmetry group define discrete Painlevé equations with the same number of parameters as their family of surfaces. While non-translation elements of the symmetry group have been observed to correspond to discrete systems of Painlevé-type through projective reduction, these have fewer than the maximal number of free parameters corresponding to their surface type. In this poster, I will report research showing that difference equations with the full number of free parameters can be constructed from non-translation elements of infinite order in the symmetry group, constructing several examples and demonstrating their integrability. This is prompted by the study of a previously proposed discrete Painlevé equation related to a special class of discrete analogues of surfaces of constant negative Gaussian curvature, which we generalise to a full-parameter integrable difference equation, given by the Cremona action of a non-translation element of the extended affine Weyl group $\widetilde{W}(D_4^{(1)})$ on a family of generic $D_4^{(1)}$- surfaces.