

# REPRESENTATION THEOREM FOR ENRICHED CATEGORIES LYNE MOSER

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# I. MOTIVATION

- **Universal properties** are useful to make many constructions in mathematics

Ex: (co)limits, adjunctions, Kan extensions, ...

- Encoded by the **representability** of a certain presheaf

Ex:  $G: I \rightarrow \mathcal{C}$  has a limit  $L \in \mathcal{C} \iff \mathcal{C}(-, L) \cong \mathcal{C}^I(\Delta(-), G)$ .

- **Not always convenient** to work with.

Ex: in  $\infty$ -category theory, homotopy type theory.

- Looking for an **internal** characterization.

Ex: Limit of a functor  $G \rightsquigarrow$  terminal object in the category of cones  $\Delta \downarrow G$ .

## II. 1-CATEGORICAL STORY

### $\mathcal{C}$ -category

A presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$  is **representable** if there is:

- an object  $L \in \mathcal{C}$ , and
- a natural isomorphism  $\mathcal{C}(-, L) \cong F$  ( $\in \text{Set}^{\mathcal{C}^{\text{op}}}$ )

### Examples:

- **Limit:**  $G: I \rightarrow \mathcal{C}$ -functor  
 $F := \mathcal{C}^I(\Delta(-), G): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ ,  $C \mapsto \mathcal{C}^I(\Delta C, G)$   
set of cones  
 $G$  has a limit  $\iff F$  is representable.
- **Adjunction:**  $L: \mathcal{C} \rightarrow \mathcal{D}$ -functor  
For  $D \in \mathcal{D}$ ,  $F_D := \mathcal{D}(L(-), D): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ ,  $C \mapsto \mathcal{D}(LC, D)$   
 $L$  has a right adjoint  $\iff F_D$  is representable  $\forall D \in \mathcal{D}$ .

Grothendieck construction:  $\int_{\mathcal{C}}: \text{Set}^{\mathcal{C}^{\text{op}}} \rightarrow \text{Cat}/\mathcal{C}$

$F: \mathcal{C}^{\text{op}} \rightarrow \text{Set} \mapsto \int_{\mathcal{C}} F \rightarrow \mathcal{C}$

where  $\int_{\mathcal{C}} F$  is the **category of elements** of  $F$

- obj: pairs  $(C, x)$  of  $C \in \mathcal{C}$ ,  $x \in FC$

- mor  $(C, x) \rightarrow (D, y)$ :  $C \xrightarrow{f} D \in \mathcal{C}$  st  $Ff(y) = x$  ( $FD \xrightarrow{Ff} FC$ )

Remark: It corresponds to the category:

$$\coprod_{C \in \mathcal{C}} FC \xrightleftharpoons[\text{t}]{\text{s}} \coprod_{C, D \in \mathcal{C}} \mathcal{C}(C, D) \times FD \xleftarrow{\text{c}} \dots \in \text{Set}$$

Example: **Limit**:  $G: I \rightarrow \mathcal{C}$ ;  $F = \mathcal{C}^I(\Delta(-), G)$

$\int_{\mathcal{C}} F \cong \Delta \downarrow G$  is the **category of cones** over  $G$

- obj: pairs  $(C, \kappa)$  of  $C \in \mathcal{C}$ ,  $\kappa: \Delta C \Rightarrow G$

- mor  $(C, \kappa) \rightarrow (D, \mu)$ :  $C \xrightarrow{f} D \in \mathcal{C}$  st  $\Delta C \xrightarrow{\Delta f} \Delta D$   
 $\kappa \Downarrow G \Leftarrow \mu$

A functor  $P: \mathcal{A} \rightarrow \mathcal{C}$  is a **discrete fibration** if for all  $A \in \mathcal{A}$  and  $C \xrightarrow{g} PA \in \mathcal{C}$ , there is a unique  $B \xrightarrow{f} A \in \mathcal{A}$  st  $Pf = g$

Remark: This is equivalent to saying that

$$\begin{array}{ccc} \text{mor } \mathcal{A} & \xrightarrow{P} & \text{mor } \mathcal{C} \\ \downarrow t & \lrcorner & \downarrow t \\ \text{ob } \mathcal{A} & \xrightarrow{P} & \text{ob } \mathcal{C} \end{array} \quad \text{is a pullback in Set}$$

Example: **Grothendieck construction**

$F: \mathcal{C}^{\mathcal{C}} \rightarrow \text{Set}$ ,  $\int_{\mathcal{C}} F \rightarrow \mathcal{C}$  is a discrete fibration

Notation:  $\text{Fib}/\mathcal{C} \subseteq \text{Cat}/\mathcal{C}$  full subcategory spanned by the discrete fibrations

Theorem: The Grothendieck construction

$\int_{\mathcal{C}}: \text{Set}^{\mathcal{C}^{\mathcal{C}}} \rightarrow \text{Fib}/\mathcal{C}$   
is an **equivalence** of categories.

Representation theorem: A presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$  is **representable**  $\Leftrightarrow \int_{\mathcal{C}} F$  has a **terminal object**.

Example: Limit:  $G: I \rightarrow \mathcal{C}$ ,  $F := \mathcal{C}^I(\Delta(-), G)$

$G$  has a limit  $\Leftrightarrow F$  is representable  
 $\Leftrightarrow \int_{\mathcal{C}} F = \Delta \downarrow G$  has a terminal object

Remark: An object  $T$  in a category  $\mathcal{C}$  is **terminal**  $\Leftrightarrow$  the projection  $\mathcal{C}/_T \rightarrow \mathcal{C}$  is an isomorphism of categories.

$$\begin{array}{ccc}
 \mathcal{C}/_T & \xrightarrow{\quad} & \mathcal{C}^2 \\
 \downarrow & \lrcorner & \downarrow \\
 \mathcal{C} & \xrightarrow{(\text{id}, \{T\})} & \mathcal{C} \times \mathcal{C}
 \end{array} \in \text{Cat}$$

### III PROBLEMS IN HIGHER DIMENSIONS

A **2-category**  $\mathcal{C}$  is a category enriched in  $\text{Cat}$ , i.e., it consists of:

- a set of objects  $\text{ob}\mathcal{C}$
- a **category** of morphisms  $\mathcal{C}(C, D)$ ,  $\forall C, D \in \mathcal{C}$
- + associative and unital compositions

$$C \rightarrow D + C \begin{array}{c} \Downarrow \\ \rightarrow \end{array} D$$

Example: **Cat** - 2-category of categories, functors, and natural transformations

$\mathcal{C}$ -2-category

A **2-presheaf** on  $\mathcal{C}$  is a 2-functor  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ .

Example: **Representable** 2-presheaf: for  $L \in \mathcal{C}$ ,

$$\mathcal{C}(-, L): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}, C \mapsto \mathcal{C}(C, L)$$

A 2-presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$  is **representable** if there is:

- an object  $L \in \mathcal{C}$
- a 2-natural isomorphism  $\mathcal{C}(-, L) \cong F$  ( $\in \mathcal{2}\text{Cat}(\mathcal{C}^{\text{op}}, \text{Cat})$ )

Example: **2-limit**:  $G: I \rightarrow \mathcal{C}$  - 2-functor

$F := \mathcal{C}^I(\Delta(-), G): \mathcal{C}^{\text{op}} \rightarrow \text{Cat}, C \mapsto \mathcal{C}^I(\Delta C, G) \leftarrow$   
category of 2-cones & modifications

$G$  has a 2-limit  $\iff F$  is representable

$L \in \mathcal{C}$  - 2-limit of  $G$ ; then  $\forall C \in \mathcal{C}$

$$\mathcal{C}(C, L) \xrightarrow{\cong} \mathcal{C}^I(\Delta C, G) \in \text{Cat}$$

$$C \rightarrow L \xleftrightarrow{\sim} \Delta C \Rightarrow G \quad (= \Delta C \Rightarrow \Delta L \Rightarrow G)$$

$$C \Downarrow L \xleftrightarrow{\sim} \Delta C \Downarrow G \quad (= \Delta C \Downarrow \Delta L \Rightarrow G)$$

An object  $T$  in a 2-category  $\mathcal{C}$  is **2-terminal** if  $\mathcal{C}(C, T) \cong \mathbb{1} \quad \forall C \in \mathcal{C}$



Try 1:  $\int_{\mathcal{C}} : 2\text{Cat}(\mathcal{C}^{\text{op}}, \text{Cat}) \rightarrow 2\text{Cat}/_{\mathcal{C}}, F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat} \mapsto \int_{\mathcal{C}} F \rightarrow \mathcal{C}$

where  $\int_{\mathcal{C}} F$  is the (strict) 2-category of elements of  $F$

- obj: pairs  $(C, x)$  of  $C \in \mathcal{C}, x \in FC$
- mor  $(C, x) \rightarrow (D, y): C \xrightarrow{f} D \in \mathcal{C}$  st  $Ff(y) = x$
- 2-mor  $(C, x) \begin{matrix} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{matrix} (D, y): C \begin{matrix} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{matrix} D \in \mathcal{C}$  st  $(F\alpha)_y = \text{id}_x$  ( $Ff \xRightarrow{F\alpha} Fg$ )

⚠ Does **not** see the morphisms of  $FC$

E.g. 2-limit of  $G \xleftarrow{\times} \xRightarrow{\quad} \mathcal{C}$  2-terminal object in the (strict) 2-category of cones  $\Delta \downarrow G = \int_{\mathcal{C}} \mathcal{C}^{\mathcal{I}}(\Delta(-), G)$ .

- obj: pairs  $(C, \kappa)$  of  $C \in \mathcal{C}, \kappa: \Delta C \Rightarrow G$
  - mor  $(C, \kappa) \rightarrow (D, \mu): C \xrightarrow{f} D \in \mathcal{C}$  st  $\begin{matrix} \Delta C & \xrightarrow{\Delta f} & \Delta D \\ \kappa \searrow & & \swarrow \mu \end{matrix}$
  - 2-mor  $(C, \kappa) \begin{matrix} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{matrix} (D, \mu): C \begin{matrix} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{matrix} D \in \mathcal{C}$  st  $\begin{matrix} \Delta C & \xrightarrow{\Delta f} & \Delta D \\ \kappa \searrow & \Downarrow \alpha & \swarrow \mu \\ & \Delta g & \end{matrix}$
- [dingman-M.]: counterexamples.

# IV SOLUTION: DOUBLE CATEGORIES

A **double category**  $\mathbb{A}$  is an internal category to categories

$$\mathbb{A}_0 \begin{matrix} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{matrix} \mathbb{A}_1 \xleftarrow{c} \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \in \text{Cat}$$

category of

• objects:  $C, D, C', D', \dots$

• vertical mor:  $C \begin{matrix} \downarrow \\ \# \\ \downarrow \end{matrix} C'$

category of

• horizontal mor:  $C \rightarrow D$

• squares:  $C \begin{matrix} \rightarrow & & D \\ \downarrow & \Downarrow & \downarrow \\ C' & \rightarrow & D' \end{matrix}$

There is a (horizontal) embedding

$$\text{Int}: 2\text{Cat} \rightarrow \text{DbCat}, \quad \mathcal{C} \mapsto \text{ob } \mathcal{C} \begin{matrix} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{matrix} \coprod_{C, D \in \mathcal{C}} \mathcal{C}(C, D)$$

Sees  $\mathcal{C}$  as a double cat. with **trivial** vertical morphisms

$$C \begin{matrix} \rightarrow & & D \\ \Downarrow & & \Downarrow \\ C & \rightarrow & D \end{matrix} \rightsquigarrow \begin{matrix} C & \rightarrow & D \\ \# & \Downarrow & \# \\ C & \rightarrow & D \end{matrix}$$

# [Paré] Double Grothendieck construction

$$\mathbb{I}_e: 2\text{Cat}(\mathcal{C}^{\varphi}, \text{Cat}) \rightarrow \text{DbCat}/\text{Inte}$$

$F: \mathcal{C}^{\varphi} \rightarrow \text{Cat} \mapsto \mathbb{I}_e F \rightarrow \text{Int}\mathcal{C}$  where  $\mathbb{I}_e F$  has:

Where  $\mathbb{I}_e F$  is the **double category of elements** of  $F$

- obj: pairs  $(C, x)$  of  $C \in \mathcal{C}, x \in FC$

- hor mor  $(C, x) \rightarrow (D, y): C \xrightarrow{f} D \in \mathcal{C}$  st  $Ff(y) = x$

- ver mor  $(C, x) \rightarrow (C, x'): x \xrightarrow{\varphi} x' \in FC$

- sq  $(C, x) \xrightarrow{f} (D, y) : C \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} D \in \mathcal{C}$  st  $x = Ff(y) \xrightarrow{Ff(\varphi)} Ff(y')$   
 $\begin{array}{ccc} \varphi \downarrow & \Downarrow & \downarrow \psi \\ (C, x') & \xrightarrow{g} & (D, y') \end{array}$   
 $\begin{array}{ccc} (F\alpha)_y \downarrow & \searrow \varphi & \downarrow (F\alpha)_{y'} \\ Fg(y) & \xrightarrow{Fg(\psi)} & Fg(y') = x' \end{array}$

Remark: It corresponds to the double category:

$$\bigsqcup_{C \in \mathcal{C}} FC \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{c} \\ \xleftarrow{t} \end{array} \bigsqcup_{C, D \in \mathcal{C}} \mathcal{C}(C, D) \times FD \xleftarrow{c} \dots \in \text{Cat}$$

[Paré] An object  $T$  in a double category  $\mathbb{A}$  is **double terminal** if

$$- \forall A \in \mathbb{A}, \exists! A \rightarrow T \in \mathbb{A} \quad - \forall \begin{array}{c} A \\ \downarrow \\ B \end{array} \in \mathbb{A}, \exists! \begin{array}{ccc} A & \xrightarrow{\quad} & T \\ \downarrow & \Downarrow & \downarrow \\ B & \xrightarrow{\quad} & T \end{array} \in \mathbb{A}$$

Remark: This is equivalent to requiring that the projection  $\mathbb{A}_{/T} \rightarrow \mathbb{A}$  is an isomorphism of double categories.

$$\begin{array}{ccc} \mathbb{A}_{/T} & \xrightarrow{\quad} & \mathbb{A}^{\text{Int} 2} \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{A} & \xrightarrow{(id, \{T\})} & \mathbb{A} \times \mathbb{A} \end{array} \in \text{DblCat}$$

Theorem: [dingman-M.] A 2-presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$  is representable  $\Leftrightarrow \int_{\mathcal{C}} F$  has a double terminal object.

Example: **2-limits**:  $G: I \rightarrow \mathcal{C}$ ,  $F := \mathcal{C}^I(\Delta(-), G)$

$G$  has a 2-limit  $\Leftrightarrow$  the double category of cones  $\Delta \Downarrow G = \int_{\mathcal{C}} F$  has a double terminal object

$$- \text{ver mor } (C, \kappa) \rightarrow (C, \kappa'): \Delta C \begin{array}{c} \xrightarrow{\kappa} \\ \Downarrow \\ \xrightarrow{\kappa'} \end{array} G$$

# V. CASE OF ENRICHED CATEGORIES

$(\mathcal{V}, x, 1)$ -cartesian closed category with pullbacks.

Need  $\mathcal{V}$  to be **extensive**, i.e., it has all small coproducts that "behave well"

Consequence:  $\text{Set} \hookrightarrow \mathcal{V}$ ,  $S \mapsto \bigsqcup_S 1$

A  **$\mathcal{V}$ -enriched category**  $\mathcal{C}$  consists of:

- a set of objects
- a hom-object  $\mathcal{C}(C, D) \in \mathcal{V} \quad \forall C, D \in \mathcal{C}$

+ associative and unital compositions

Examples:  $\mathcal{V} = \text{Set} \rightsquigarrow \text{Cat}$ ;  $\mathcal{V} = \text{Cat} \rightsquigarrow 2\text{Cat}$

Example:  $\mathcal{V}$  is a  $\mathcal{V}$ -enriched category with  
- hom objects the internal homs  $Y^X$ , for  $X, Y \in \mathcal{V}$ .

Remark:  $\text{Set} \hookrightarrow \mathcal{V}$  induces an inclusion  $\text{Cat} \hookrightarrow \mathcal{V}\text{-Cat}$   
 $\mathcal{C}$ - $\mathcal{V}$ -enriched category

A  $\mathcal{V}$ -presheaf on  $\mathcal{C}$  is a  $\mathcal{V}$ -functor  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$

Example: **Representable**  $\mathcal{V}$ -presheaf: for  $L \in \mathcal{C}$ ,  
 $\mathcal{C}(-, L): \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}, C \mapsto \mathcal{C}(C, L)$ .

A  $\mathcal{V}$ -presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$  is **representable** if there is:

- an object  $L \in \mathcal{C}$
- a  $\mathcal{V}$ -natural isomorphism  $\mathcal{C}(-, L) \cong F$  ( $\in \mathcal{V}\text{Cat}(\mathcal{C}^{\text{op}}, \mathcal{V})$ )

Example:  **$\mathcal{V}$ -enriched limits**:  $G: \mathbf{I} \rightarrow \mathcal{C}$ - $\mathcal{V}$ -functor

$F := \mathcal{C}^{\mathbf{I}}(\Delta(-), G): \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}, C \mapsto \mathcal{C}^{\mathbf{I}}(\Delta C, G)$   $\leftarrow$   
object of  $\mathcal{V}$ -cones in  $\mathcal{V}$

$G$  has a  $\mathcal{V}$ -enriched limit  $\Leftrightarrow F$  is representable.

An **internal category to  $\mathcal{V}$**   $\mathbb{A}$  is a diagram

$$A_0 \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{c} \\ \xleftarrow{t} \end{array} A_1 \xleftarrow{c} A_1 \times_{A_0} A_1 \in \mathcal{V}$$

+ source and target relations  
+ associativity and unitality of composition.

Examples:  $\mathcal{V} = \text{Set} \rightsquigarrow \text{Cat}$ ;  $\mathcal{V} = \text{Cat} \rightsquigarrow \text{DbCat}$

There is an embedding

$$\text{Int}: \mathcal{V}\text{-Cat} \rightarrow \text{Cat}(\mathcal{V}), \mathcal{C} \mapsto \text{ob}\mathcal{C} \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{c} \\ \xleftarrow{t} \end{array} \bigsqcup_{C, D \in \mathcal{C}} \mathcal{C}(C, D)$$

Sees  $\mathcal{C}$  as an internal category with discrete 0<sup>th</sup> level

Grothendieck construction:

$$\int_{\mathcal{C}}: \mathcal{V}\text{-Cat}(\mathcal{C}^{\text{op}}, \mathcal{V}) \rightarrow \text{Cat}(\mathcal{V})_{/\text{Int}\mathcal{C}}, F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{V} \mapsto \int_{\mathcal{C}} F \rightarrow \text{Int}\mathcal{C}$$

where  $\int_{\mathcal{C}} F$  is the internal category to  $\mathcal{V}$

$$\bigsqcup_{C \in \mathcal{C}} FC \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{c} \\ \xleftarrow{t} \end{array} \bigsqcup_{C, D \in \mathcal{C}} \mathcal{C}(C, D) \times FC \xleftarrow{c} \dots \in \mathcal{V}$$

An internal functor  $P: \mathbb{A} \rightarrow \text{Int}\mathbb{C}$  in  $\text{Cat}(\mathcal{V})$  is a **discrete fibration** if

$$\begin{array}{ccc} \mathbb{A}_1 & \xrightarrow{P} & (\text{Int}\mathbb{C})_1 = \coprod_{C, D \in \mathbb{C}} \mathbb{C}(C, D) \\ t \downarrow \perp & & \downarrow t \\ \mathbb{A}_0 & \xrightarrow{P} & (\text{Int}\mathbb{C})_0 = \text{ob}\mathbb{C} \end{array}$$

is a pullback in  $\mathcal{V}$

Example: **Grothendieck construction**

$F: \mathbb{C}^{\varphi} \rightarrow \mathcal{V}$ ,  $\int_e F \rightarrow \text{Int}\mathbb{C}$  is a discrete fibration

Notation:  $\text{Fib}/_e \subseteq \text{Cat}(\mathcal{V}) / \text{Int}\mathbb{C}$  full subcategory spanned by the discrete fibrations over  $\text{Int}\mathbb{C}$

Theorem: [M.-Sanzóla-Verdugo]

The Grothendieck construction

$$\int_e: \mathcal{V}\text{-Cat}(\mathbb{C}^{\varphi}, \mathcal{V}) \rightarrow \text{Fib}/_e$$

is an **equivalence** of categories.



An object  $T$  in an internal category  $\mathbb{A}$  to  $\mathcal{V}$  is **terminal** if  $\mathbb{A}_{/T} \rightarrow \mathbb{A}$  is an isomorphism in  $\text{Cat}(\mathcal{V})$ .

$$\begin{array}{ccc}
 \mathbb{A}_{/T} & \longrightarrow & \mathbb{A}^{\text{Int} \mathcal{V}} \\
 \downarrow \lrcorner & & \downarrow \\
 \mathbb{A} & \xrightarrow{(\text{id}, \{T\})} & \mathbb{A} \times \mathbb{A}
 \end{array} \in \text{Cat}(\mathcal{V})$$

Theorem: [M.-Sanzola-Verdugo]

A  $\mathcal{V}$ -presheaf  $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{V}$  is representable  $\iff \int_{\mathcal{C}} F$  has a terminal object.

Example:  $\mathcal{V}$ -limits:  $G: \mathbb{I} \rightarrow \mathcal{C}$ ,  $F := \mathcal{C}^{\mathbb{I}}(\Delta(-), G)$

$G$  has a  $\mathcal{V}$ -limit  $\iff$  the internal category of  $\mathcal{V}$ -cones  $\Delta \downarrow G := \int_{\mathcal{C}} F$  has a terminal object.

Application:  $\mathcal{V} = (n-1)\text{Cat}$

$(n-1)\text{Cat} - \text{Cat} = n\text{Cat}$ ;  $\text{Cat}((n-1)\text{Cat}) \neq n\text{foldCat}$

$n$ -limits  $\iff$  terminal object in an internal category to  $(n-1)$ -categories of  $n$ -cones