The geometry of coherent topoi & ultrastructures

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YAMCATS
Dec 2022, Manchester.

This talk is based on a preprint that you can find on the ArXiv.

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Conceptual completeness

Let $f : \mathcal{F} \to \mathcal{G}$ be a morphisms of pretopoi. If the induced functor between categories of models is an equivalence of categories, then f is an equivalence too,

$$f^* : \mathsf{Mod}(\mathcal{G}) \to \mathsf{Mod}(\mathcal{F}).$$

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Idea: let ultrastructures emerge as a necessary structure so that we can isolate the correct definition.

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So, for example, for an essentially algebraic theory $\mathbb{T}$, $pt(\mathcal{E}_{\mathbb{T}})$ is complete and cocomplete.

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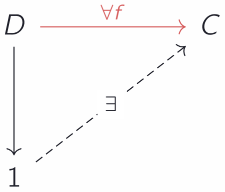

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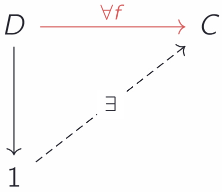

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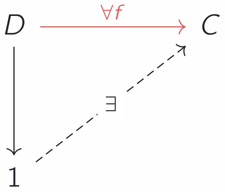

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In the recent paper **KZ monads and Kan Injectivity** by Sousa, Lobbia and DL this behaviour is called Weak Kan Injectivity.

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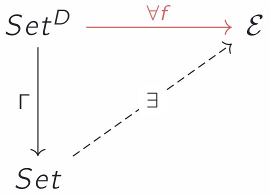

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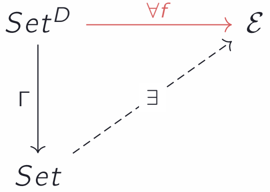

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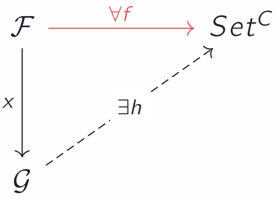

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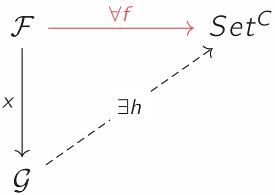

Define $h^* = \mathrm{lan}_y(x_* f^* y)$. One can show that in this case $h_* = \mathrm{lan}_{x_*}(f_*)$.

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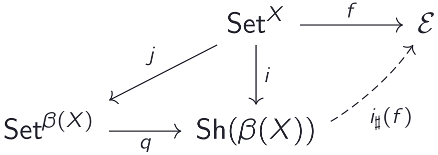
Now, consider a coherent topos and recall that we are Kan injective with respect to i.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$$\begin{tikzcd} & \mathsf{Set}^X \arrow[r, "f"] \arrow[dl, "j"'] \arrow[d, "i"] & \mathcal{E} \\ \mathsf{Set}^{\beta(X)} \arrow[r, "q"'] & \mathsf{Sh}(\beta(X)) \arrow[ur, dashed, "i_\sharp(f)"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{pt}(\mathcal{E})^X \arrow[d, dash, "\simeq"'] & \mathsf{Topoi}(\mathsf{Sh}(\beta(X)), \mathcal{E}) \arrow[r, "q^\sharp"] & \mathsf{Topoi}(\mathsf{Set}^{\beta(X)}, \mathcal{E}) \arrow[d, dash, "\simeq"'] \\ \mathsf{Topoi}(\mathsf{Set}, \mathcal{E})^X \arrow[r, dash, "\simeq"'] & \mathsf{Topoi}(\mathsf{Set}^X, \mathcal{E}) \arrow[u, "i_\sharp"] & \mathrm{pt}(\mathcal{E})^{\beta(X)} \end{tikzcd}$$

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