School of Computer Science
University of Birmingham

Adelic Geometry via Topos Theory

(joint work with Steve Vickers)

Ming Ng
mxn732@bham.ac.uk

September 22, 2022

What this talk is about

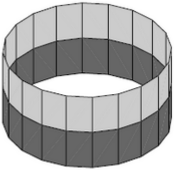
I'm going to discuss two basic themes that cross-cut many different areas of mathematics:

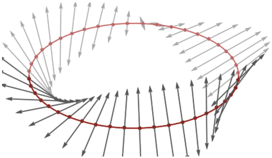
1. What kind of info can topological data encode?
2. When can we solve a problem by breaking it into smaller pieces?

I'll then discuss how the research project 'Adelic Geometry via Topos Theory' serves as an interesting test problem for illuminating how these two themes interact with each other.

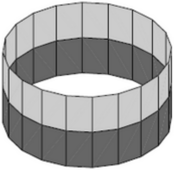
Homotopical Data = Geometric Data

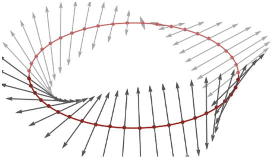
Homotopical Data = Geometric Data





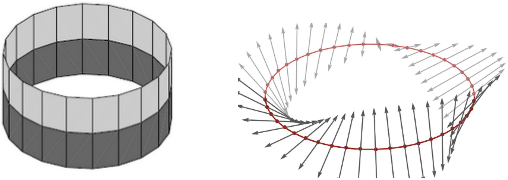
Homotopical Data = Geometric Data





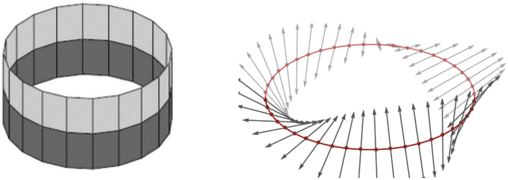
- Question: Are these the only line bundles over S^1 (up to isomorphism)?

Homotopical Data = Geometric Data



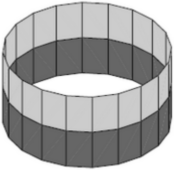
- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.

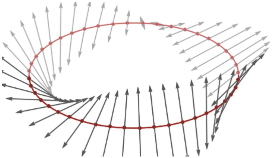
Homotopical Data = Geometric Data



- Question: Are these the only line bundles over S^1 (up to isomorphism)?
- Answer: Yes.
- Why? Exploit the tight relationship between $[S^{k-1}, GL_n(\mathbb{R})]$ and $\mathrm{Vect}_n(S^k)$.

Homotopical Data = Geometric Data





Classification Theorem

Suppose that X is a paracompact space. Let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of n-dimensional vector bundles over X. Then the map

$$[X, G_n] \longrightarrow \mathrm{Vect}_n(X)$$

given by $f \longmapsto f^*(\gamma_n)$ is a bijection, where γ_n is the universal bundle.

Continuity = Geometricity

A similar attitude occurs in topos theory in regards to geometric logic:

> In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto *"continuity is geometricity"*. In other words, to "do mathematics continuously" is to work within the geometricity constraints.

— Vickers, 'Continuity and Geometric Logic'

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups);

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is a structure that 'satisfies' these axioms.

Geometric Logic

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is a structure that 'satisfies' these axioms.

Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives

Theories and Models

A *theory* can be viewed as an axiomatic description of mathematical structures (e.g. the theory of groups); a *model* is a structure that 'satisfies' these axioms.

Geometric Theory

A *geometric theory* is a theory whose (formulae featured in its) axioms are built out of certain logical connectives — i.e. $=$, **finite** conjunctions $\wedge$, **arbitrary** (possibly infinite) disjunctions $\bigvee$, and $\exists$.

Example: Theory of Dedekind Reals

As an example, consider the geometric theory of Dedekind reals, which we denote $\mathbb{R}$.

Example: Theory of Dedekind Reals

As an example, consider the geometric theory of Dedekind reals, which we denote $\mathbb{R}$. A model x of $\mathbb{R}$ is a Dedekind real number, which will be represented by two sets of rationals (L, R), whereby:

$$L = \{q \in \mathbb{Q} | q < x\}$$

$$R = \{r \in \mathbb{Q} | x < r\}$$

Otherwise known as the left and right Dedekind sections of the real number.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves **finite** limits and **arbitrary** colimits.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves **finite** limits and **arbitrary** colimits.

Points of a Topos

Definition

- A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ of toposes is a pair of adjoint functors $f_* : \mathcal{F} \to \mathcal{E}$ and $f^* : \mathcal{E} \to \mathcal{F}$, respectively called the *direct image* and the *inverse image* of f, such that the left adjoint f^* preserves **finite** limits and **arbitrary** colimits.

Definition

1. A *global point* of a topos $\mathcal{E}$ is defined as a geometric morphism $\mathrm{Set} \to \mathcal{E}$.
2. A *generalised point* of a topos $\mathcal{E}$ is a geometric morphism $\mathcal{F} \to \mathcal{E}$.

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\textbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathrm{mod}(\mathcal{E})$$

Theorem

Every Grothendieck topos is a classifying topos of some geometric theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ has a classifying topos.

Topos = Generalised Space

Definition

The classifying topos of a geometric theory $\mathbb{T}$ is a Grothendieck topos $\mathrm{Set}[\mathbb{T}]$ that classifies the models of $\mathbb{T}$ in Grothendieck toposes, i.e. for any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories:

$$\mathbf{Geom}(\mathcal{E}, \mathrm{Set}[\mathbb{T}]) \simeq \mathbb{T}\text{-mod}(\mathcal{E})$$

Theorem

Every Grothendieck topos is a classifying topos of some geometric theory $\mathbb{T}$, and every geometric theory $\mathbb{T}$ has a classifying topos.

Slogan

Models = points of a topos. In particular, we can reason in terms of the points of the topos (as a generalised space) as opposed to only reasoning in terms of its objects/sheaves (as a category).

Point-free Topology - A Bird's Eye View

Point-set Topology

- Point = Element of a set
- Space = A set of points, along with a set of opens satisfying some specific axioms.
- Continuous Maps = A function $f : X \to Y$ such that $f^{-1}(U)$ is open for all opens $U \subset Y$

Pointfree Topology

- Point = Model of a geometric theory
- Space = The 'World' in which the point lives with other points (i.e. a Grothendieck topos)
- Continuous Maps = A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ preserves finite limits and small colimits

Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos,

Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos, which possesses the universal property that that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained as $f^*(U_{\mathbb{T}}) \cong M$ via the inverse image functor of some (unique) $f : \mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

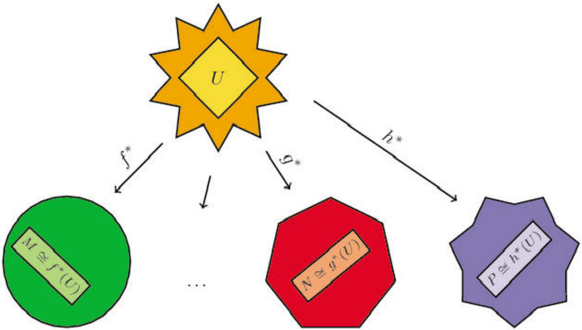
Generic Model

Fact

There exists a *generic model* $U_{\mathbb{T}}$ living in every classifying topos, which possesses the universal property that that any model M in a Grothendieck topos $\mathcal{E}$ can be obtained as $f^*(U_{\mathbb{T}}) \cong M$ via the inverse image functor of some (unique) $f : \mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

An **important** consequence of this is that any geometric sequent that holds for $U_{\mathbb{T}}$ will hold for all models M of $\mathbb{T}$.

Generic Model



Classifying topos

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial?

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

‘Divide-and-Conquer’

$$X^n + Y^n + Z^n = 0 \qquad (n > 2)$$

- Question: What are the rational (equiv. integer) solutions to this polynomial? — hard!
- Observation #1: Integer solutions imply real and modulo p solutions (in fact p-adic solutions).
- Observation #2: Real and p-adic solutions are easier to deal with than just integer/rational solutions.

- New Question: Given a polynomial with $\mathbb{Q}$-coefficients, when does knowledge about its $\mathbb{Q}_p$ and $\mathbb{R}$-solutions give us info about its $\mathbb{Q}$-solutions?

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

Hasse's Local-Global Principle

Local-Global Principle for $\mathbb{Q}$

Some property P is true for $\mathbb{Q}$ iff P is true for all the completions of $\mathbb{Q}$.

Definition of adele ring for $\mathbb{Q}$

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is defined to be the restricted product of all the completions of $\mathbb{Q}$. Morally, the adele ring can be viewed as a device that allows us to reason about all the completions of $\mathbb{Q}$ simultaneously.

Idea

Instead of asking whether a property simultaneously holds for *all completions* of $\mathbb{Q}$ (which forces us to use complicated algebraic constructions like the adele ring $\mathbb{A}_{\mathbb{Q}}$), what if we asked whether a property holds for the *generic completion* of $\mathbb{Q}$?

Generic Completion

"One weakness in the analogy between the collection of $\{K_s\}_{s \in S}$ for a compact Riemann surface S and the collection $\{\mathbb{Q}_p$, for prime numbers p, and $\mathbb{R}\}$ is that [...] no manner of squinting seems to be able to make $\mathbb{R}$ the least bit mistake-able for any of the p-adic fields, nor are the p-adic fields $\mathbb{Q}_p$ isomorphic for distinct p.

***A major theme in the development of Number Theory has been to try to bring $\mathbb{R}$ somewhat more into line with the p-adic fields;** a major mystery is why $\mathbb{R}$ resists this attempt so strenuously."*

— Mazur, 'Passage from Local to Global in Number Theory'

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$

Investigation begins...

Starting point:

For simplicity, let us assume that our base field is $\mathbb{Q}$. Classically, an absolute value of $\mathbb{Q}$ is a function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y \in \mathbb{Q}$:

- $|x| \geq 0$, and $|x| = 0$ iff $x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$

We define a *place* as an equivalence class of absolute values whereby $|\cdot|_1 \sim |\cdot|_2$ if there exists some $\alpha \in (0, 1]$ such that $|\cdot|_1 = |\cdot|_2^\alpha$ or $|\cdot|_2 = |\cdot|_1^\alpha$.

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:

  1. $|\cdot|^\alpha \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0, 1]$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^\alpha \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0, 1]$

$$[av] \times (0,1] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^\alpha \sim |\cdot|$

  for any absolute value $|\cdot|$, and $\alpha \in (0,1]$

$$[av] \times (0,1] \underset{ex}{\overset{\pi}{\rightrightarrows}} [av]$$

- π is the projection map sending $(|\cdot|, \alpha) \mapsto |\cdot|$
- *ex* is the exponentiation map sending $(|\cdot|, \alpha) \mapsto |\cdot|^\alpha$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^\alpha \sim |\cdot|$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^\alpha \sim |\cdot|$
  2. $|\cdot|^1 = |\cdot|$
  3. $(|\cdot|^\alpha)^\beta = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha \cdot \beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:
  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2022)

Classifying Topos of Places of $\mathbb{Q}$

- Intuitively: what does this topos look like?
- The points of this topos would correspond to equivalence classes of absolute values, such that:
  1. $|\cdot|^{\alpha} \sim |\cdot|$
  2. $|\cdot|^{1} = |\cdot|$
  3. $(|\cdot|^{\alpha})^{\beta} = |\cdot|^{\alpha\cdot\beta}$

  for any absolute value $|\cdot|$, and $\alpha, \beta \in (0, 1]$
- In essence, we would like to 'quotient' the topos [*av*] by an algebraic action – two questions:
  - Is the notion of (real) exponentiation geometric? Ng-Vickers (2022)
  - What does it mean to quotient by a monoid action vs. group action?

Global vs. Local Picture

Ostrowski's Theorem for $\mathbb{Q}$

Every absolute value of $\mathbb{Q}$ is equivalent to a (non-Archimedean) p-adic absolute value $|\cdot|_p$ (for some prime p), or the Archimedean absolute value $|\cdot|_\infty$.

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}[column sep=large] {[av_{NA}] \times (0,\infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D} \end{tikzcd}$$

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}
{[av_{NA}] \times (0,\infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}
{[av_{NA}] \times (0,\infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^\alpha$ still yields a non-Arch. absolute value for any $\alpha \in (0,\infty)$ (unlike the Archimedean case).

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}
{[av_{NA}] \times (0,\infty)} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^{\alpha}$ still yields a non-Arch. absolute value for any $\alpha \in (0,\infty)$ (unlike the Archimedean case).
- What is $\mathcal{D}$?

Non-Archimedean Place (for fixed prime p)

$$\begin{tikzcd}[av_{NA}] \times (0,\infty) \arrow[r, bend left, "\pi"] \arrow[r, bend right, "ex"'] & {[av_{NA}]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}\end{tikzcd}$$

- For any non-Arch. absolute $|\cdot|$, exponentiating $|\cdot|^\alpha$ still yields a non-Arch. absolute value for any $\alpha \in (0,\infty)$ (unlike the Archimedean case).
- What is $\mathcal{D}$?

Theorem

$\mathcal{D} \simeq \{*\}$

Archimedean Place

$$
\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}
$$

Archimedean Place

$$\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0, 1]$-action as opposed to a **group** $(0, \infty)$-action.

Archimedean Place

$$\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case?

Archimedean Place

$$\begin{tikzcd}[av_A] \times (0,1] \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'\end{tikzcd}$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0,1]$-action as opposed to a **group** $(0,\infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No!

Archimedean Place

$$\begin{tikzcd}
{[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}'
\end{tikzcd}$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0, 1]$-action as opposed to a **group** $(0, \infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos corresponding to $\mathcal{D}'$ has non-trivial forking in its sheaves)

Archimedean Place

$$\begin{tikzcd} {[av_A] \times (0,1]} \arrow[r, bend left, "\pi"] \arrow[r, bend right, "\mathfrak{m}"'] & {[av_A]} \arrow[l, "s"] \arrow[r, dashed] & \mathcal{D}' \end{tikzcd}$$

- Space of Arch. absolute values is acted upon by a **monoid** $(0, 1]$-action as opposed to a **group** $(0, \infty)$-action.
- Can we play the same game as we did in the Non-Archimedean case? Answer: No! (The topos corresponding to $\mathcal{D}'$ has non-trivial forking in its sheaves)
- So what is $\mathcal{D}'$?

Archimedean Place

Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$

Archimedean Place

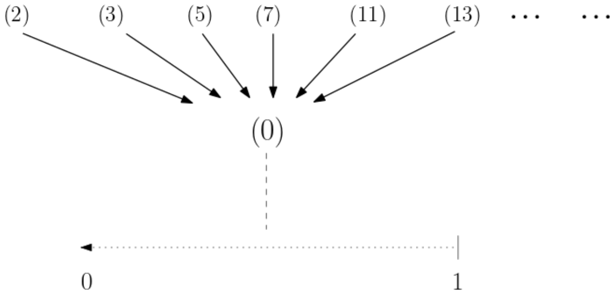
Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

Archimedean Place

Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)



Archimedean Place

Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

Archimedean Place

Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ...

Archimedean Place

Theorem

$\mathcal{D}' \simeq \overleftarrow{[0,1]}$ (the space of 'upper reals' between 0 and 1)

- The Arakelov compactification of $\mathrm{Spec}(\mathbb{Z})$ suggests that we add a single point at infinity to $\mathrm{Spec}(\mathbb{Z})$ corresponding to the 'Archimedean prime' ... our picture suggests that there is some blurring going on at infinity, and that infinity is not just a classical point with no intrinsic structure.

A Strange Woods

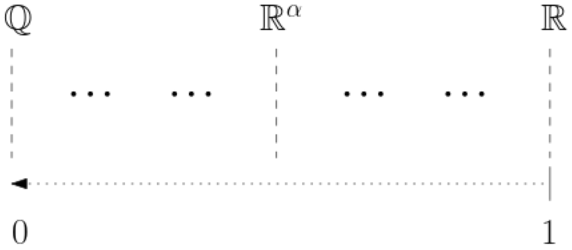
Function Field Analogy

*"In order to re-establish the **analogy**, it is necessary to introduce, into the theory of algebraic numbers, something that corresponds to **the point at infinity** in the theory of functions [...] to define a "**prime ideal at infinity**" [...]*

If one follows it in all of its consequences, the theory alone permits us to reestablish the analogy at many points where it once seemed defective."

— André Weil's Letter to his sister, 1940

Function Field Analogy



Reorienting our perspective

The issue of how to unite the Archimedean and the non-Archimedean settings is not (just) an algebraic question, but a topological one: how should the connected and the disconnected interact?

Topos-Theoretic Implications

- Caramello: “Toposes as Unifying Bridges”

$$
\begin{tikzcd}
 & \color{red}{\mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'}} \arrow[dr, dashed, bend left] & \\
\mathbb{T} \arrow[ur, dashed, bend left] & & \mathbb{T}'
\end{tikzcd}
$$

Topos-Theoretic Implications

- Caramello: “Toposes as Unifying Bridges”

$$
\begin{tikzcd}
 & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed] & \\
\mathbb{T} \arrow[ur, dashed] & & \mathbb{T}'
\end{tikzcd}
$$

*“Toposes can effectively act as **unifying spaces** for **transferring information** between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.”*

— Olivia Caramello

Topos-Theoretic Implications

- Caramello: “Toposes as Unifying Bridges”

$$\begin{tikzcd}[column sep=small] & \mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{T}'} \arrow[dr, dashed, bend left] & \\ \mathbb{T} \arrow[ur, dashed, bend left] & & \mathbb{T}' \end{tikzcd}$$

A Basic Challenge

When is this a helpful framework for transferring info? When is it not so helpful?

Logical Berkovich Geometry

Question: where else might we see spaces whose points correspond to valuations?

Question: where else might we see spaces whose points correspond to valuations?

Berkovich Affine Line

Let K be a non-Archimedean field that is non-trivially valued.

Logical Berkovich Geometry

Question: where else might we see spaces whose points correspond to valuations?

Berkovich Affine Line

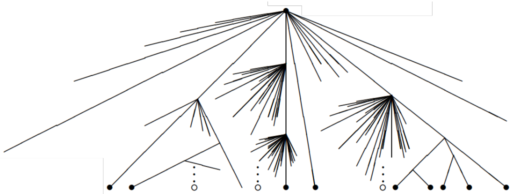
Let K be a non-Archimedean field that is non-trivially valued.The **Berkovich Affine line** $\mathbb{A}^1_{\mathrm{Berk}}$ is the set of multiplicative seminorms on $K[T]$ (extending $|\cdot|$ on K), equipped with the Berkovich topology.

Logical Berkovich Geometry

Question: where else might we see spaces whose points correspond to valuations?

Berkovich Affine Line

Let K be a non-Archimedean field that is non-trivially valued.The **Berkovich Affine line** $\mathbb{A}^1_{\mathrm{Berk}}$ is the set of multiplicative seminorms on $K[T]$ (extending $|\cdot|$ on K), equipped with the Berkovich topology.



Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assuming that K is non-Arch. + non-trivially valued:

Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assuming that K is non-Arch. + non-trivially valued:

1. The set of bounded multiplicative seminorms on $K[T]$ equipped with the Berkovich topology;

Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assuming that K is non-Arch. + non-trivially valued:

1. The set of bounded multiplicative seminorms on $K[T]$ equipped with the Berkovich topology;
2. A space whose points are defined by a sequence of nested closed discs $D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \dots$ contained in K;
3. The space of types over K, concentrating on $\mathbb{A}^1_K$, that are "almost orthogonal to Γ";
4. A profinite $\mathbb{R}$-tree.

Logical Berkovich Geometry

Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assuming that K is non-Arch. + non-trivially valued:

1. The set of bounded multiplicative seminorms on $K[T]$ equipped with the Berkovich topology;
2. A space whose points are defined by a sequence of nested closed discs $D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \dots$ contained in K;
3. The space of types over K, concentrating on $\mathbb{A}^1_K$, that are "almost orthogonal to Γ";
4. A profinite $\mathbb{R}$-tree.

Theorem (N.)

Let $\mathcal{A} := K\{R^{-1}T\}$ be the ring of formal power series convergent in radius $R > 0$, where K is non-Arch but need not be non-trivially valued.

Logical Berkovich Geometry

Equivalent Characterisations of $\mathbb{A}^1_{\mathrm{Berk}}$

Assuming that K is non-Arch. + non-trivially valued:

1. The set of bounded multiplicative seminorms on $K[T]$ equipped with the Berkovich topology;
2. A space whose points are defined by a sequence of nested closed discs $D_{r_1}(k_1) \supseteq D_{r_2}(k_2) \supseteq \ldots$ contained in K;
3. The space of types over K, concentrating on $\mathbb{A}^1_K$, that are "almost orthogonal to Γ";
4. A profinite $\mathbb{R}$-tree.

Theorem (N.)

Let $\mathcal{A} := K\{R^{-1}T\}$ be the ring of formal power series convergent in radius $R > 0$, where K is non-Arch but need not be non-trivially valued. Then, the space of R-good filters is (classically) equivalent to the Berkovich spectrum $\mathcal{M}(\mathcal{A})$.

By way of conclusion...

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.

By way of conclusion...

- Theme #1: Viewing toposes as a framework uniting logic and topology
- Theme #2: Local-Global issues, and its connections to Theme #1 via generic reasoning

- Pulling away from the set theory reveals key insights into the deep nerve connecting topology and algebra.
- Some very interesting indications that there is some blurring at infinity in our picture of $\overline{\mathrm{Spec}(\mathbb{Z})}$ — interesting to explore the precise implications of this + broader question of how the connected and disconnected ought to interact.

References I

[1] Balchin, S. and Greenlees, J.P.C. *Adelic models of tensor-triangulated categories*, Advances in Mathematics, 375:107339, (2020).

[2] Berkovich, V.I., *Spectral theory and analytic geometry over non-Archimedean fields, Mathematical Surveys and Monographs*, Vol. 33, American Mathematical Society, Providence, RI, (1990)

[3] Boileau, A., Joyal, A. *La Logique des Topos*, The Journal of Symbolic Logic, Vol. 46, No. 1, pp. 6-16, (1981).

[4] Caramello, O., *Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges'*, Oxford University Press (2017).
Volume 162, pp. 518-551, May 2016.

References II

[5] Egan, G., *Geometry and Waves [Extra]*, `http://www.gregegan.net/ORTHOGONAL/03/WavesExtra.html`

[6] Hatcher, A.: *Vector Bundles and K-theory*, https://www. math.cornell.edu/ hatcher/VBKT/VBpage.html

[7] Hrushovski, E., Loeser, F., *Non-Archimedean Tame Topology and Stably Dominated Types*, (AM-192), Princeton University Press (2016).

[8] Johnstone, P.T.: *Topos Theory*, Dover Publications Inc., (1977)

[9] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 1*, Clarendon Press, (2002)

[10] Johnstone, P.T.: *Sketches of an Elephant: A Topos Theory Compendium, Vol. 2*, Clarendon Press, (2002)

References III

[11] Mazur, B., *On the Passage from Local to Global in Number Theory*, Bulletin of the AMS, 29 No. 1, (1993).

[12] Moerdijk, I. *The classifying topos of a continuous groupoid, I.*, Transactions of the American Mathematical Society Volume 310, Number 2, pp. 629-668, 1988.

[13] Nlab, *Vector Bundles*, `https://ncatlab.org/nlab/show/vector+bundle`.

[14] Ng, M., and Vickers, S. *Point-free Construction of Real Exponentiation*, Logical Methods of Computer Science (2022)

[15] Vickers, S., *Localic Completion of Generalized Metric Spaces*, preprint.

[16] Vickers, S., *Continuity and Geometric Logic*, J. Applied Logic (12), pp. 14-27, (2014).

References IV

[17] Weil, A., Letter to Sister on Analogy in Mathematics (1940), translated by Martin Krieger, Doing Mathematics, (2003), World Scientific Publishing.

[18] Zakharevich, I., *Atttiudes of K-theory*, *NOTICES OF THE AMERICAN MATHEMATICAL SOCIETY*, Vol. 66, No. 7, pp. 1034-1044.