Topological quantum field theories & homotopy cobordisms

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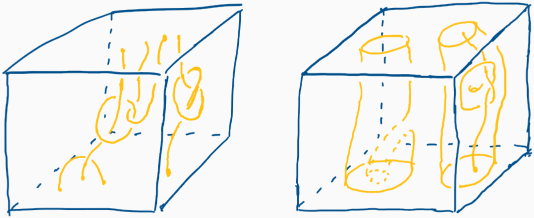

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- Such functors may factor through other categories that may be easier to work with - I will give a construction of a category of *cofibrant cospans* of topological spaces. Functors into this category are obtained roughly by taking the complement of particle trajectories.

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- Such functors may factor through other categories that may be easier to work with - I will give a construction of a category of *cofibrant cospans* of topological spaces. Functors into this category are obtained roughly by taking the complement of particle trajectories.
- I will also show that a Yetter's TQFTs associated to finite groups generalise to functors from this category.

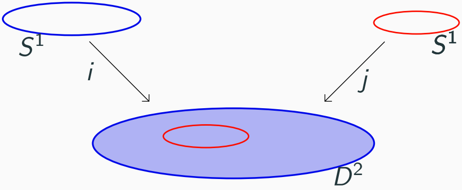
Cofibrant cospans and homotopy cobordisms

Cofibrant cospans

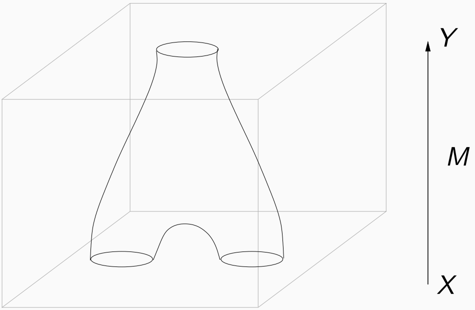
Definition
Let X, Y and M be spaces. A cofibrant cospan from X to Y is a diagram $i\colon X \to M \leftarrow Y : j$ such that $\langle i,j\rangle\colon X \sqcup Y \to M$ is a closed cofibration.
For spaces $X, Y \in \mathbf{Top}$, we define the set of all concrete cofibrations

$$\mathrm{CofCos}(X,Y) = \left\{ \begin{matrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{matrix} \;\middle|\; \langle i,j\rangle \text{ is a closed cofibration} \right\}.$$

Cofibrant cospans



Cofibrant cospans



Example
Let X be a space. The cospan $\mathrm{id}_X\colon X \to X \leftarrow X\colon \mathrm{id}_X$ is not a cofibrant cospan, unless $X = \varnothing$.

Proposition
For X a topological space, the cospan $\iota_0^X\colon X \to X \times \mathbb{I} \leftarrow X\colon \iota_1^X$ is a cofibrant cospan (where $\iota_a^X\colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x, a)$).

Composition of cofibrant cospans

Lemma

(I) For any spaces X, Y and Z in $Ob(\mathbf{Top})$ there is a composition of cofibrant cospans

$$\bullet : \mathrm{CofCos}(X, Y) \times \mathrm{CofCos}(Y, Z) \to \mathrm{CofCos}(X, Z)$$

$$\left(\begin{array}{ccc} X & & Y \\ & {}_{i}\searrow \; M \; \swarrow_{j} & \end{array} , \begin{array}{ccc} Y & & Z \\ & {}_{k}\searrow \; N \; \swarrow_{l} & \end{array} \right) \mapsto \begin{array}{ccc} X & & Z \\ & {}_{\tilde{i}}\searrow \; M \sqcup_Y N \; \swarrow_{\tilde{l}} & \end{array}$$

where $\tilde{i} = p_M \circ i$ and $\tilde{l} = p_N \circ l$ are obtained via the following diagram

$$\begin{array}{ccccccc} X & & & Y & & & Z \\ & {}_{i}\searrow & \swarrow_{j} & & {}_{k}\searrow & \swarrow_{l} & \\ & & M & & N & & \\ & & & {}_{p_M}\searrow \quad \swarrow_{p_N} & & & \\ & & & M \sqcup_Y N, & & & \end{array}$$

the middle square of which is the pushout of $j \colon M \leftarrow Y \rightarrow N \colon k$ in $\mathbf{Top}$.

Equivalence classes cofibrant cospans

Lemma

For each pair $X, Y \in Ob(\mathrm{CofCos})$, we define a relation on $\mathrm{CofCos}(X, Y)$ by

$$\begin{pmatrix} X & & Y \\ & {}_{i}\searrow \; M \; \swarrow_{j} & \end{pmatrix} \overset{ch}{\sim} \begin{pmatrix} X & & Y \\ & {}_{i'}\searrow \; N \; \swarrow_{j'} & \end{pmatrix}$$

if there exists a commuting diagram

$$\begin{array}{ccccc} & & M & & \\ & {}^{i}\nearrow & & \nwarrow^{j} & \\ X & & \downarrow \psi & & Y \\ & {}_{i'}\searrow & & \swarrow_{j'} & \\ & & M' & & \end{array}$$

where ψ is a homotopy equivalence. For each pair $X, Y \in \textbf{Top}$ the relations $(\mathrm{CofCos}(X, Y), \overset{ch}{\sim})$ are a congruence on CofCos.

Category of cofibrant cospans

Theorem (T.)
The quadruple

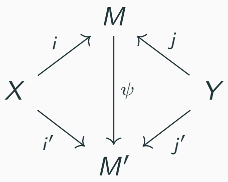
$$\mathrm{CofCos} = \left(Ob(\mathbf{Top})\ ,\ \mathsf{CofCos}(X,Y)/\overset{ch}{\sim}\ ,\ \bullet\ ,\ \left[\begin{array}{ccc} X & & X \\ & {\scriptstyle\iota_0^X}\searrow \quad \swarrow{\scriptstyle\iota_1^X} & \\ & X \times \mathbb{I} & \end{array} \right]_{\mathsf{ch}} \right)$$

is a category.

Category of cofibrant cospans

Proof uses classical theorem (E.g. Brown06, Thm7.2.8):

If $X \xrightarrow{i} M \xleftarrow{j} Y$, $X \xrightarrow{i'} N \xleftarrow{j'} Y$ are cospans such that $\langle i, j\rangle \colon X \sqcup Y \to M$ and $\langle i', j'\rangle \colon X \sqcup Y \to N$ are cofibrations, then the set of homotopy equivalences ψ such that



commutes, is in bijective correspondence with the set of ψ' such that there exists $\phi \colon N \to M$ with $\psi' \circ \phi$ and $\phi \circ \psi'$ homotopic to identity through maps commuting with cospans.

Monoidal category of cofibrant cospans

There is a functor $\Phi\colon \mathbf{Top} \to \mathrm{CofCos}$ which sends a continuous map $f\colon X \to Y$ to the cospan $\begin{array}{ccc} X & & Y \\ & {}_{\iota_0^Y \circ f}\searrow \quad \swarrow_{\iota_1^Y} & \\ & Y \times \mathbb{I} & \end{array}$.

Theorem (T.)
There is a monoidal category $(\mathrm{CofCos}, \otimes, \varnothing, \alpha_{X,Y,Z}, \lambda_X, \rho_X, \beta_{X,Y})$ where

$$\begin{bmatrix} W & & X \\ & {}_i\searrow \quad \swarrow_j & \\ & M & \end{bmatrix}_{\mathrm{ch}} \otimes \begin{bmatrix} Y & & Z \\ & {}_k\searrow \quad \swarrow_l & \\ & N & \end{bmatrix}_{\mathrm{ch}} = \begin{bmatrix} W \sqcup Y & & X \sqcup Z \\ & {}_{i \sqcup k}\searrow \quad \swarrow_{j \sqcup l} & \\ & M \sqcup N & \end{bmatrix}_{\mathrm{ch}}.$$

All other maps are the images of the corresponding maps in $(\mathbf{Top}, \sqcup)$.

Definition
A space X is called *homotopically* 1*-finitely generated* if $\pi(X, A)$ is finitely generated for all finite sets of basepoints A.
Let χ denote the class of all homotopically 1-finitely generated spaces.

Theorem
There is a (symmetric monoidal) subcategory of CofCos

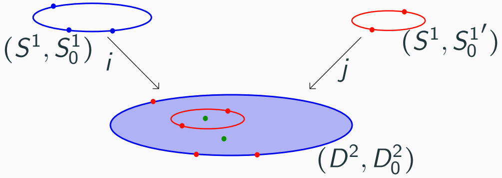
$$\mathrm{HomCob} = \left(\chi, \mathrm{HomCob}(X, Y), \cdot, \left[\begin{array}{ccc} X & & X \\ & {\scriptstyle \iota_0^X} \searrow \quad \swarrow {\scriptstyle \iota_1^X} & \\ & X \times \mathbb{I} & \end{array} \right]_{\mathsf{ch}} \right).$$

$Z_G \colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Definition
Let (X, X_0), (Y, Y_0) and (M, M_0) be such that X, Y and M are in χ and X_0, Y_0, M_0 finite representative subsets. A *concrete based homotopy cobordism* from (X, X_0) to (Y, Y_0) is a diagram $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$ such that:

1. $i\colon X \to M \to Y\colon j$ is a concrete homotopy cobordism.
2. i and j are maps of pairs.
3. $M_0 \cap i(X) = i(X_0)$ and $M_0 \cap j(Y) = j(Y_0)$.

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

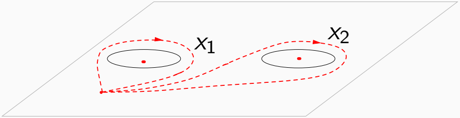


For a pair $(X, X_0) \in \chi$, define

$$Z^!_G(X, X_0) = \mathbb{C}\left(\mathbf{Grpd}\left(\pi(X, X_0), G\right)\right).$$

Example

$\pi(X, X_0) \cong (\mathbb{Z} * \mathbb{Z}) \sqcup \{*\} \sqcup \{*\}$. Maps from $\pi(X, X_0)$ to G are determined by pairs in $G \times G$, whose elements respectively denote the images of the equivalence classes of the loops marked x_1 and x_2 in the figure, so we have $\mathrm{Z}^{!}_{G}(X, X_0) \cong \mathbb{C}(G \times G)$.



$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Let $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$ be a concrete based homotopy cobordism, we define a matrix

$$Z^!_G \left(\begin{smallmatrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \ \ \swarrow_j & \\ & (M,M_0) & \end{smallmatrix} \right) : Z^!_G(X, X_0) \to Z^!_G(Y, Y_0)$$

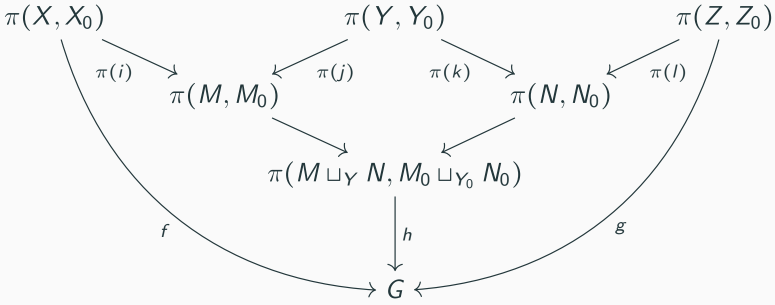
as follows. Let $f \in Z^!_G(X, X_0)$ and $g \in Z^!_G(Y, Y_0)$ be basis elements, then

$$\left\langle g \,\middle|\, Z^!_G \left(\begin{smallmatrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \ \ \swarrow_j & \\ & (M,M_0) & \end{smallmatrix} \right) \middle|\, f \right\rangle = \left| \left\{ h : \pi(M, M_0) \to G \;\middle|\; \begin{array}{ccccc} \pi(X,X_0) & & & & \pi(Y,Y_0) \\ & \overset{\pi(i)}{\searrow} & & \overset{\pi(j)}{\swarrow} & \\ & & \pi(M,M_0) & & \\ & {}_f\searrow & \downarrow h & \swarrow_g & \\ & & G & & \end{array} \right\} \right|$$

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Lemma

The map $Z^!_G$ preserves composition, extended in the obvious way to a composition of based cospans.



Lemma
Let X be a topological space, G a group, $X_0 \subseteq X$ a finite representative subset and $y \in X$ a point with with $y \notin X_0$. There is a non-canonical bijection of sets

$$\Theta_\gamma\colon \mathbf{Grpd}(\pi(X, X_0), G) \times G \to \mathbf{Grpd}(\pi(X, X_0 \cup \{y\}), G)$$
$$(f, g) \mapsto F$$

where γ is a choice of a path from some $x \in X_0$ to y and F is the extension along γ and g.

Consider a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$. It follows

$$Z^!_G(M, M_0 \cup \{m\}) = |G|\, Z^!_G(M, M_0).$$

It follows that for all M_0' and M_0, we can write

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0|)} Z^!_G(M, M_0)$$

and

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0'|)} Z^!_G(M, M_0')$$

which together imply

$$|G|^{-|M_0|} Z^!_G(M, M_0) = |G|^{-|M_0'|} Z^!_G(M, M_0')$$

and that

$$|G|^{-(|M_0| - |X_0|)} Z^!_G(M, M_0) = |G|^{-(|M_0'| - |X_0|)} Z^!_G(M, M_0').$$

Lemma

We redefine the linear map we assign to a concrete based homotopy cobordisms as

$$Z_G^{!!}\left(\begin{array}{ccc}(X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{array}\right) = |G|^{-(|M_0|-|X_0|)} Z_G^{!}\left(\begin{array}{ccc}(X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{array}\right).$$

The map $Z_G^{!!}$ does not depend on the choice of subset $M_0 \subseteq M$, and this preserves composition. When the relevant cospan is clear, we will refer to this as $Z_G^{!!}(M, X_0, Y_0)$ to highlight the dependence on X_0 and Y_0.

$\mathrm{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Lemma
There is a contravariant functor

$$\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$$

constructed as follows. Let $X_\alpha, X_\beta \in Ob(\mathbf{FinSet}^*(X))$ with $X_\beta \subseteq X_\alpha$. Let $\mathcal{V}_X(X_\alpha) = \mathbf{Grpd}(\pi(X, X_\alpha), G)$. For any $v_\alpha \in \mathcal{V}_X(X_\alpha)$ we have a commuting triangle

$$\begin{array}{ccc} \pi(X, X_\beta) & \xrightarrow{\iota_{\beta\alpha}} & \pi(X, X_\alpha) \\ & \searrow^{v_\alpha \circ \iota_{\beta\alpha}} & \downarrow v_\alpha \\ & & G. \end{array}$$

Now let $\mathcal{V}_X(\iota_{\beta\alpha}\colon X_\beta \to X_\alpha) = \phi_{\alpha\beta}$ where $\phi_{\alpha\beta} : \mathcal{V}_X(X_\alpha) \to \mathcal{V}_X(X_\beta)$, $v_\alpha \mapsto v_\alpha \circ \iota_{\alpha\beta}$.

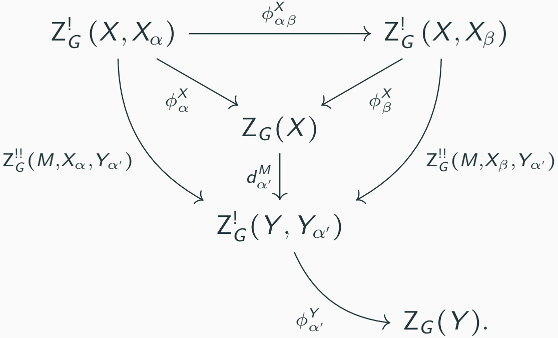
$\mathrm{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Definition
For $X \in \chi$ define

$$\mathrm{Z}_G(X) = \mathrm{colim}(\mathcal{V}'_X) = \mathbb{C}(\mathrm{colim}(\mathcal{V}_X))$$

where $\mathcal{V}'_X = F_{V_{\mathbb{C}}} \circ \mathcal{V}_X$ and $\mathcal{V}_X\colon \mathbf{FinSet}^*(X) \to \mathbf{Set}$.

Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism. Fix a choice of $Y_{\alpha'} \subseteq Y$ such that $(Y, Y_{\alpha'}) \in \chi$. For each pair $X_\alpha, X_\beta \subseteq X$ such that $(X, X_\alpha), (X, X_\beta) \in \chi$ we have the following diagram



(1)

Lemma
The assignment

$$Z_G\begin{pmatrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{pmatrix} = \phi^{Y}_{\alpha'} d^{M}_{\alpha'}$$

does not depend on the choice of $Y_{\alpha'}$.

Theorem (T.)
Z_G is a functor.

Lemma
Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in Z_G(X)$ and $[g] \in Z_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $colim(\mathcal{V}_X)$), then

$$\langle [g] | Z_G(M) | [f] \rangle = |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \mid h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right|$$

$$= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, Z_G^{!!}(M, M_0) \,\middle|\, f \right\rangle$$

where $\phi_0^Y\colon Z_G^!(Y, Y_0) \to Z_G(Y)$ is the map into $colim(\mathcal{V}_Y')$.

Lemma

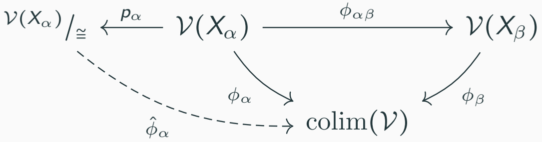
Let $i\colon X \to M \leftarrow Y :j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$ a choice of concrete based homotopy cobordism, and $[f] \in Z_G(X)$ and $[g] \in Z_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $colim(\mathcal{V}_X)$), then

$$\begin{aligned}\langle [g] | Z_G(M) | [f] \rangle &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \mid h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right| \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, Z_G^{!!}(M, M_0) \,\middle|\, f \right\rangle\end{aligned}$$

where $\phi_0^Y\colon Z_G^!(Y, Y_0) \to Z_G(Y)$ is the map into $colim(\mathcal{V}'_Y)$. Equivalently

$$\langle [g] | Z_G(M) | [f] \rangle = |G|^{-(|M_0|-|X_0|)} \left| \left\{ h : \pi(M, M_0) \to G \mid h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} \sim g \right\} \right|$$

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$



Theorem (T.)

For X a space, the map $\hat{\phi}_\alpha$ is an isomorphism. Hence, for a homotopically 1-finitely generated space $X \in \chi$

$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X, X_0), G)/\cong),$$

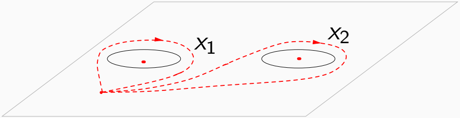
for any choice $X_0 \subset X$ of finite representative subset, where $\cong$ denotes taking maps up to natural transformation.

Further,

$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X), G)/\cong),$$

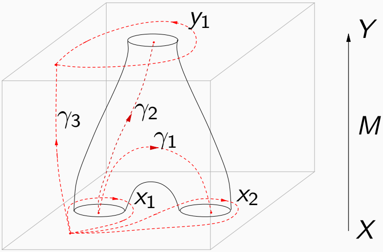
.

Example



Let X be the complement of the embedding of two circles shown. Letting $X_0 \subset X$ be the subset shown, $\textbf{Grpd}(\pi(X, X_0), G) = G \times G$ as discussed previously. Since all objects are mapped to the unique object in G, taking maps up to natural transformation is means taking maps up to conjugation by elements of G at each basepoint, hence in this case maps are labelled by pairs of elements of G, up to simultaneous conjugation, so we have $Z_G(X) = \mathbb{C}((G \times G)/G)$.

Example



Basis elements in $Z_G(X)$ are given by equivalence classes $[(f_1, f_2)]$ where $f_1, f_2 \in G$ and $[]$ denotes simultaneous conjugation by the same element of G.
Basis elements in $Z_G(Y)$ are given by elements of g taken up to conjugation, denoted $[g_1]$. We have

$$\begin{aligned}\langle [g_1] | Z_G(M) | [(f_1, f_2)] \rangle &= |G|^{-2} \left\{ a, b, c, d, e \in G \mid a = f_1, b = f_2, g_1 \sim ebae^{-1} \right\} \\ &= \left\{ e \in G \mid g_1 \sim ef_1 f_2 e^{-1} \right\} \\ &= \begin{cases} |G| & \text{if } g_1 \sim f_1 f_2 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Example

Let $x \in X$ be the basepoint which is in the connected component of X homotopy equivalent to the punctured disk, and $x' \in X$ some choice of basepoint in the other connected component. There is a bijection sending a map $h \in \mathbf{Grpd}(\pi(M, M_0), G)$ to a quadruple $(h', h(\gamma_1), h(\gamma_2), h(\gamma_3)) \in \mathbf{Grpd}(\pi(M, \{x, x'\}) \times G \times G \times G$, where h' is the restriction of h to $\pi(M, \{x, x'\})$. Now $\pi(M, \{x, x'\})$ is the disjoint union of the groupoids $\pi(M_1, \{x\})$ and $\pi(M_2, \{x'\})$ where M_1 is the path connected component of M containing x, and M_2 is the path connected component containing x'. The group $\pi(M_2, \{x'\})$ is trivial, so there is one unique map into G. The group $\pi(M_1, \{x\})$ is isomorphic to the twice punctured disk, which has fundamental group isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. This isomorphism can be realised by sending the loop x_1 to the 1 in the first copy of $\mathbb{Z}$ and x_2 to the 1 in the second copy of $\mathbb{Z}$. Thus we can label elements in $\mathbf{Grpd}(\pi(M_1, \{x\}), G)$ by elements of $G \times G$ where $g_1 \in (g_1, g_2)$ corresponds to the image of x_1, and g_2 the image of x_2. Hence a map in $\mathbf{Grpd}(\pi(M, M_0), G)$ is determined by a five tuple $(a, b, c, d, e) \in G \times G \times G \times G \times G$ where a corresponds to the image of x_1, b to the image of x_2, and c, d and e correspond to the images of γ_1, γ_2 and γ_3 respectively.