Yorkshire & Midlands Category Theory Seminar
26^{th} meeting

Dialectica completion & dialectica logical principles

based on a joint work with Davide Trotta (University of Pisa) *&* Valeria de Paiva (Topos Institute)

speaker Matteo Spadetto (University of Leeds)

Gödel's Dialectica Interpretation

Dialectica Interpretation is based on a theory, called System T, in a many-sorted language $\mathcal{L}$ and such that any formula of T is quantifier free. Whenever A is a formula in the language of arithmetic, then we inductively define a formula A^D in the language $\mathcal{L}$ of the form $\exists x.\forall y.A_D$, where A_D is quantifier free. This interpretation satisfies the following:

Theorem

If HA proves a formula A, then T proves $A_D(t, y)$ where t is a sequence of closed terms.

Dialectica construction

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An **object** of $\mathrm{Dial}(\mathcal{C})$ is a triple (X, U, α), which we think of as a formula $(\exists x)(\forall u)\alpha(x, u)$, where α is a subobject of $X \times U$ in $\mathcal{C}$.

Dialectica construction

An **arrow** from $(\exists x)(\forall u)\alpha(x,u)$ to $(\exists y)(\forall v)\beta(y,v)$ is a pair $(F\colon X \longrightarrow Y,\ f\colon X \times V \longrightarrow U)$, i.e. a pair $(F(x) : Y,\ f(x,v) : U)$ of terms in context satisfying the condition $\alpha(x, f(x,v)) \leq \beta(F(x), v)$ between the reindexed subobjects, where the squares:

$$\begin{array}{ccc} \alpha(x,f(x,v)) & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[\langle \mathrm{pr}_X, f\rangle]{} & X \times U \end{array} \qquad \begin{array}{ccc} \beta(F(x),v) & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[F\times 1_V]{} & Y \times V \end{array}$$

are pullbacks.

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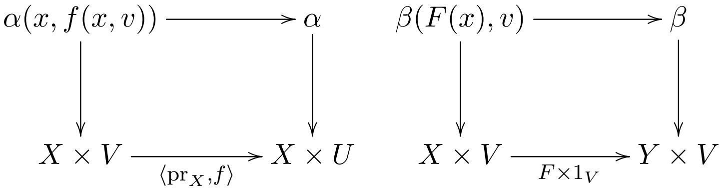

are pullbacks.

The notion of morphism of $\mathrm{Dial}(\mathcal{C})$ is motivated by the definition of the dialectica interpretation for formulas of the form $A \to B$:

$$(A \to B)^D = \exists F.\exists f.\forall x.\forall v.(\ A_D(x, f(x,v)) \to B_D(F(x), v)\ ).$$

Re-indexing in a cloven and split fibration

Let $\mathcal{C}$ be a category with finite products and let $p \colon \mathcal{E} \to \mathcal{C}$ be a Grothendieck fibration.

We think of $\mathcal{C}$ as the category of **contexts** associated to a given type theory. Whenever A is an object of $\mathcal{C}$, then the objects α of $\mathcal{E}_A$ represent **predicates** $\alpha(a)$ in context $a \colon A$ and the arrows $\alpha \to \beta$ of $\mathcal{E}_A$ represent **proofs** of $\beta(a)$ from $\alpha(a)$ in context $a \colon A$. Let $B \xrightarrow{f} A$ be an arrow of $\mathcal{C}$, i.e. a (finite list of) **terms in context** $b : B \mid f(b) : A$. The reindexing $f^* \colon \mathcal{E}_A \to \mathcal{E}_B$ via $f(b)$ is defined as follows:

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$$\begin{CD} \alpha(f(b)) @>>> \alpha \\ @V{g(f(b))}VV @VV{g}V \\ \alpha'(f(b)) @>>> \alpha'. \end{CD}$$

Existential fibrations

A fibration p is **existential** if $\mathrm{pr}^* \colon \mathcal{E}_A \to \mathcal{E}_{A \times B}$ has a left adjoint $\exists_{\mathrm{pr}} \colon \mathcal{E}_{A \times B} \to \mathcal{E}_A$ for any projection $A \times B \xrightarrow{\mathrm{pr}} A$ of the base category (satisfying the BC condition).

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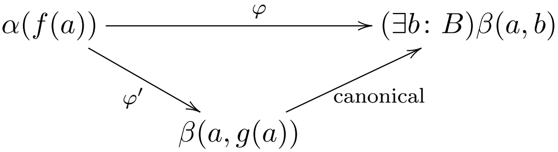
Let $p \colon \mathcal{E} \to \mathcal{C}$ be an existential fibration. We say that a predicate $\alpha(i)$ in $\mathcal{E}_I$ is **∃-free** if it enjoys the following universal property:

for every arrow $A \xrightarrow{f} I$ of $\mathcal{C}$ and every arrow:

$$\alpha(f(a)) \xrightarrow{\varphi} (\exists b \colon B)\beta(a,b)$$

of $\mathcal{E}_A$, where $\beta(a,b)$ is a predicate in $\mathcal{E}_{A\times B}$, there exist a unique arrow $A \xrightarrow{g} B$ and a unique arrow $\alpha(f(a)) \xrightarrow{\varphi'} \beta(a,g(a))$ of $\mathcal{E}_A$ such that:



commutes.

Gödel fibrations

Dually there is a notion of **universal** fibration and **∀-free** predicate.

Definition

Let $\mathcal{C}$ be a cartesian closed category and let $p\colon \mathcal{E} \to \mathcal{C}$ be a fibration. We say that p is a **Gödel fibration** if:

1. the fibration p is existential;
2. the fibration p **has enough ∃-free** predicates, that is, for every object A in $\mathcal{C}$ and every predicate α in $\mathcal{E}_A$, there is an ∃-free predicate β in some $\mathcal{E}_{A\times B}$ such that $\alpha \cong (\exists b\colon B)\beta(a,b)$;
3. the full subfibration p' of ∃-free predicates of p is universal;
4. the fibration p' has enough ∀-free predicates.

The ∀-free predicates of p' are called the **quantifier-free** predicates of p.

Gödel fibration

Proposition (Prenex normal form)

If a fibration $p\colon \mathcal{E} \to \mathcal{C}$ *is a Gödel fibration, then, for any predicate* α *in* $\mathcal{E}_A$*, it is the case that:*

$$\alpha(a) \cong (\exists x\colon X)(\forall y\colon Y)\beta(x, y, a)$$

where β *is a quantifier-free predicate in* $\mathcal{E}_{X\times Y\times A}$*.*

Theorem (Skolemisation)

If a fibration $p\colon \mathcal{E} \to \mathcal{C}$ *is a Gödel fibration, then, for any predicate* β *in* $\mathcal{E}_{X\times Y\times A}$*, it is the case that:*

$$(\forall x\colon X)(\exists y\colon Y)\beta(x, y, a) \cong (\exists f\colon Y^X)(\forall x\colon X)\beta(x, \mathrm{ev}(f, x), a).$$

Gödel fibrations

Theorem

If a fibration $p\colon \mathcal{E} \to \mathcal{C}$ is a Gödel fibration and α and β are quantifier-free predicates in $\mathcal{E}_{A\times X\times U}$ and in $\mathcal{E}_{A\times Y\times V}$ respectively, then an arrow:

$$(\exists x)(\forall u)\alpha(a,x,u) \to (\exists y)(\forall v)\beta(a,y,v)$$

is a triple:

$$(A\times X \xrightarrow{F} Y,\ \ A\times X\times V \xrightarrow{f} U,\ \varphi)$$

such that:

$$\alpha(a,x,f(a,x,v)) \xrightarrow{\varphi} \beta(a,F(a,x),v)$$

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$$(\textit{ that means } \langle \mathrm{pr}_{A\times X}, f\rangle^{*}\alpha \xrightarrow{\varphi} (\langle \mathrm{pr}_A, F\rangle \times 1_V)^{*}\beta\ ).$$

(A presentation of) the dialectica completion

Let $p \colon \mathcal{E} \to \mathcal{C}$ be a fibration. The **dialectica fibration** $\mathrm{Dial}(p) \colon \mathrm{Dial}(\mathcal{E}) \to \mathcal{C}$ associated to p is defined as follows:

- the **objects** of $\mathrm{Dial}(\mathcal{E})$ are quadruples (A, X, U, α) where A, X and U are objects of $\mathcal{C}$ and $\alpha \in \mathcal{E}_{A \times X \times U}$;
- an **arrow** $(A, X, U, \alpha) \to (B, Y, V, \beta)$ is a quadruple

$$(A \xrightarrow{g} B,\ A \times X \xrightarrow{F} Y,\ \ A \times X \times V \xrightarrow{f} U,\ \varphi)$$

where:

$$\alpha(a, x, f(a, x, v)) \xrightarrow{\varphi} \beta(g(a), F(a, x), v)$$

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  $$(A \xrightarrow{g} B,\ A \times X \xrightarrow{F} Y,\ A \times X \times V \xrightarrow{f} U,\ \varphi)$$

  where:

  $$\alpha(a, x, f(a, x, v)) \xrightarrow{\varphi} \beta(g(a), F(a, x), v)$$

  is an arrow in $\mathcal{E}_{A\times X\times V}$.

Then $\mathrm{Dial}(p)$ is the projection on the first component.

Theorem (Hofstra, 2011)

There is an isomorphism of fibrations:

$$\mathrm{Dial}(p) \cong \mathrm{Ex}(\mathrm{Un}(p))$$

which is natural in p.

We can use this result in order to answer some questions:

- Given a fibration p, when is it the case that there is p' such that $\mathrm{Dial}(p') = p$?
- In this case, what does p' look like?
- Which logical principles does $\mathrm{Dial}(p')$ verify?
- Which fragment of first-order logic does Dial preserve?

Characterisation of the dialectica completion

Theorem

A fibration $p\colon \mathcal{E} \to \mathcal{C}$ is a dialectica completion if and only if:

1. *the fibration p is existential;*
2. *the fibration p has enough $\exists$-free predicates;*
3. *the full subfibration p' of $\exists$-free predicates of p is universal;*
4. *the fibration p' has enough $\forall$-free predicates.*

Let p'' be the full subfibration of p' whose predicates are the $\forall$-free predicates of p' (which me might call quantifier-free*) predicates. Then* $\mathrm{Dial}(p'') \cong p$.

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Corollary

If $\mathcal{C}$ is cartesian closed, then the dialectica completions $\mathcal{E} \to \mathcal{C}$ are precisely the Gödel fibrations. Moreover, these are both existential and universal (Hofstra).

Dialectica principles

Suppose that a given fibration $p\colon \mathcal{E} \to \mathcal{C}$ is both a Heyting fibration and a Gödel fibration. Then:

Theorem

The fibration p satisfies the **Rule of Independence of Premise**, *i.e. whenever $\beta \in \mathcal{E}_{A\times B}$ and $\alpha \in \mathcal{E}_A$ is a existential-free predicate such that:*

$$a : A \mid \top \vdash \alpha(a) \to (\exists b)\beta(a,b)$$

it is the case that $a : A \mid \top \vdash (\exists b)(\alpha(a) \to \beta(a,b))$.

Theorem

The fibration p satisfies the following **Modified Markov's Rule**, *i.e. whenever $\beta_D \in \mathcal{E}_A$ is a quantifier-free predicate and $\alpha \in \mathcal{E}_{A\times B}$ is an existential-free predicate such that:*

$$a : A \mid \top \vdash (\forall b)\alpha(a,b) \to \beta_D(a)$$

it is the case that $a : A \mid \top \vdash (\exists b)(\alpha(a,b) \to \beta_D(a))$.

Preservation of logical structures

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Proposition

Let $p\colon \mathcal{E} \to \mathcal{C}$ *be a (posetal) fibration. If* p *has fibred finite conjunctions, then* $\mathrm{Ex}(p)\colon \mathrm{Ex}(\mathcal{E}) \to \mathcal{C}$ *has fibred finite conjunctions as well.*

Preservation of logical structures

Definition

Let $\mathcal{C}$ be a distributive category with points. We say that a (posetal) fibration $p\colon \mathcal{E} \to \mathcal{C}$ is **extendable** if it has both left and right adjoints to the reindexings along injections and fibred finite conjunctions and disjunctions.

Theorem

If $\mathcal{C}$ is cartesian closed and $p\colon \mathcal{E} \to \mathcal{C}$ is an extendable fibration, then $\mathrm{Dial}(p)\colon \mathrm{Dial}(\mathcal{E}) \to \mathcal{C}$ is an existential, universal and extendable fibration.

In particular (follows by Proposition 9.2.1 of Categorical Logic & Type Theory) *it is the case that $\mathrm{Dial}(\mathcal{E})$ has finite products and finite coproducts.*

Proof relevant setting?

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Definition (Sean Moss, PhD thesis)

Let $\mathcal{C}$ and $\mathcal{D}$ be functors and let $F\colon \mathcal{C} \longleftarrow \mathcal{D}$ and $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be functors. We say that the pair (F, G) is a **right-weak adjunction** of the categories $\mathcal{C}$ and $\mathcal{D}$ if there is a natural transformation $\mathcal{C}(F-, -) \xrightarrow{(-)^\flat} \mathcal{D}(-, G-)$ together with a choice of a section $(-)^\sharp$ of every (D, C)-component of $(-)^\flat$, being C an object of $\mathcal{C}$ and D an object of $\mathcal{D}$. We also say that F is **right-weakly left adjoint** to G and that G is **right-weakly right adjoint** to F.

Definition

Let $\mathcal{C}$ be a distributive category with points. We say that a fibration $p\colon \mathcal{E} \to \mathcal{C}$ is **weakly extendable** if it has right-weakly left and left-weakly right adjoints to the reindexings along injections and if its fibres are weakly finitely complete and weakly finitely cocomplete.

Proof-relevant setting?

Theorem

If $\mathcal{C}$ is cartesian closed and $p\colon \mathcal{E} \to \mathcal{C}$ is a weakly extendable fibration, then $\mathrm{Dial}(p)\colon \mathrm{Dial}(\mathcal{E}) \to \mathcal{C}$ *is an existential, universal and weakly extendable fibration.*
In particular it is the case that $\mathrm{Dial}(\mathcal{E})$ *has weak finite products and weak finite coproducts.*

Proof-relevant setting?

Theorem

If $\mathcal{C}$ is cartesian closed and $p\colon \mathcal{E} \to \mathcal{C}$ is a weakly extendable fibration, then $\mathrm{Dial}(p)\colon \mathrm{Dial}(\mathcal{E}) \to \mathcal{C}$ is an existential, universal and weakly extendable fibration.
In particular it is the case that $\mathrm{Dial}(\mathcal{E})$ has weak finite products and weak finite coproducts.
Moreover, if p has fibred (strong) finite products and (strong) right adjoints to the reindexings along the injections, then the fibration $\mathrm{Dial}(p)$ has fibred finite products and, hence, its total category $\mathrm{Dial}(\mathcal{E})$ has finite products.

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