Game comonads and Courcelle's theorem

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Game Comonads

Comonads $(\mathbb{C}, \varepsilon, \overline{(-)})$

- $\mathbb{C} \colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{A})$
- counit $\quad \varepsilon_A \colon \mathbb{C}A \to A \quad (\forall A \in \mathrm{obj}(\mathcal{A}))$
- lifting $\quad \overline{f} \colon \mathbb{C}A \to \mathbb{C}B \quad (\forall f \colon \mathbb{C}A \to B)$
- axioms: $\quad \overline{\varepsilon_A} = \mathrm{id}_{\mathbb{C}A}, \quad \varepsilon_B \circ \overline{f} = f, \quad \overline{g \circ \overline{f}} = \overline{g} \circ \overline{f}$

Example: Lists/words comonad in Sets

- $\mathrm{List}_k(A) = \{\, [a_1, \ldots, a_n] \mid a_i \in A,\ 1 \leq n \leq k\}$
- $\varepsilon([a_1, \ldots, a_n]) = a_n$.
- $f \colon \mathrm{List}_k(A) \to B$ lifts to

$$\overline{f} \colon\ [a_1, \ldots, a_n] \ \mapsto\ [b_1, \ldots, b_n]$$

where $b_i = f([a_1, \ldots, a_i])$

Eilenberg–Moore category of $\mathrm{CoAlg}(\mathbb{C})$

$\alpha\colon A \to \mathbb{C}A$ is a **coalgebra** iff

$$\begin{array}{ccc} A & & \\ {\scriptstyle\alpha}\downarrow & \overset{\text{id}}{\searrow} & \\ \mathbb{C}A & \xrightarrow[\varepsilon]{} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & \mathbb{C}A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta=\overline{\text{id}}} \\ \mathbb{C}A & \xrightarrow[\mathbb{C}\alpha]{} & \mathbb{C}^2A \end{array}$$

Adjunction:

$$\mathrm{CoAlg}(\mathbb{C}) \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{A}, \qquad U \dashv F$$

Example:

Coalgebras $\alpha\colon A \to \mathrm{List}_k(A)$ $\longleftrightarrow$ forest preorders $(\leq) \subseteq A \times A$ of depth $\leq k$

$$\alpha \longmapsto a \leq b \overset{def}{\equiv} \alpha(a) \sqsubseteq \alpha(b)$$

$$\alpha(a) = {\downarrow}a \longleftarrow\!\shortmid (\leq)$$

Category of relational structures

Fix a relational signature $\sigma = \langle R_1, \ldots, R_m \rangle$ (+ constants).

A σ-structure (or just structure) is a tuple $(A, R_1^A, \ldots, R_m^A)$ where

$$R_i^A \subseteq \underbrace{A \times \cdots \times A}_{\text{arity of } R_i} \qquad \text{for } i = 1, \ldots, m$$

We fix a **category** $\mathcal{R}(\sigma)$ of σ-relational structures and homomorphisms, i.e. maps

$$h\colon (A, R_1^A, \ldots, R_m^A) \to (B, R_1^B, \ldots, R_m^B)$$

such that

$$(x_1, \ldots, x_u) \in R_i^A \quad \text{implies} \quad (h(x_1), \ldots, h(x_u)) \in R_i^B$$

Example: If $\sigma = \langle R(\cdot, \cdot) \rangle$ then $\mathcal{R}(\sigma) =$ directed graphs.

Ehrenfeucht-Fraissé comonad(s) $\mathbb{E}_k \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma)$

(For fixed σ and $k \in \mathbb{N}$.)

Given $A \in \mathcal{R}(\sigma)$, define $\mathbb{E}_k A$ on $\mathrm{List}_k(A)$ with

$$(w_1, \ldots, w_u) \in R_i^{\mathbb{E}_k A} \quad \text{if} \quad w_s \sqsubseteq w_t \text{ or } w_s \sqsupseteq w_t \quad (\forall s, t)$$
$$\text{and} \quad (\varepsilon(w_1), \ldots, \varepsilon(w_u)) \in R_i^A$$

Sequence of subcomonads

$$\mathbb{E}_1 \rightarrowtail \mathbb{E}_2 \rightarrowtail \mathbb{E}_3 \rightarrowtail \mathbb{E}_4 \rightarrowtail \ldots$$

Combinatorial property: tree-depth

Proposition

There is a bijective correspondence between

- *coalgebras* $A \to \mathbb{E}_k A$ *and*
- *compatible forest orders* $(A, \sqsubseteq)$ *of depth* $\leq k$

Corollary (Abramsky–Shah 2018)

A σ*-structure admits an* $\mathbb{E}_k$*-coalgebra iff it has tree-depth* $\leq k$.

Logical fragments and Game Comonads

Define

- $A \leftrightarrows_{\mathbb{E}_k} B$ if there exist homomorphisms $\mathbb{E}_k A \to B$ and $\mathbb{E}_k B \to A$

- $J \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$, $A \mapsto (A, =)$ where $\sigma^I = \sigma \cup \{\ I(\cdot,\cdot)\ \}$
- $A \cong_{\mathsf{CoAlg}(\mathbb{E}_k)} B$ if $FJ(A) \cong FJ(B)$ in $\mathsf{CoAlg}(\mathbb{E}^I_k)$

Theorem (Abramsky–Shah 2018)

- $A \equiv_{\exists^+ FO_k} B$ *iff* $A \leftrightarrows_{\mathbb{E}_k} B$
- $A \equiv_{FO_k(\#)} B$ *iff* $A \cong_{\mathsf{CoAlg}(\mathbb{E}_k)} B$

Open pathwise-emb.

$A \leftrightarrow_{\mathbb{E}_k} B$ iff exists (a span of) open pathwise-embeddings

$$FJ(A) \leftarrow R \rightarrow FJ(B) \quad \text{in} \quad \mathsf{CoAlg}(\mathbb{E}_k^I)$$

Theorem (Abramsky–Shah 2020)

$A \equiv_{FO_k} B$ *iff* $A \leftrightarrow_{\mathbb{E}_k} B$

Other game comonads

Comonad	Combinat. property	logic	$\leftrightarrows_{\mathbb{C}}$	$\leftrightarrow_{\mathbb{C}}$	$\cong_{\mathsf{CoAlg}(\mathbb{C})}$
$\mathbb{E}_k$	tree-depth	qrank $\leq k$ fragment	✓	✓	✓
$\mathbb{P}_k$	tree-width	k-variable fragment	✓	✓	✓
$\mathbb{M}_k$	sync. tree depth	modal depth $\leq k$	✓	✓	✓
$\mathbb{PR}_k$	path-width	restricted conjunction k-variable	✓	?	✓
$\mathbb{He}_k$	k-ary generalised t.-w.	generalised quantifier k-variable extension	✓	✓	✓
$\mathbb{G}_k$	guarded tree decomp	quantifier-guarded	✓	✓	?
$\mathbb{LG}_k$	hypertree-width	k-conjunct guarded	✓	?	?
$\mathbb{Hy}_k$	generated tree-depth	hybrid modal depth	✓	✓	✓
$\mathbb{B}_k$	generated tree-depth	bounded quantifiers	✓	✓	✓

plus "reachability" versions of some

Comonadic Courcelle's Theorem

Courcelle's theorems

Theorem (Courcelle 1990)

Fix an MSO φ and $k \in \mathbb{N}$. Given G of tree-width $\leq k$, checking $G \models \varphi$ is in linear time.

Theorem (Courcelle–Makowsky 2000)

Fix an MSO φ and $k \in \mathbb{N}$. Given G of clique-width $\leq k$, checking $G \models \varphi$ is in linear time.

Uses that $G \in \langle B_1, \ldots, B_n \rangle_{\mathrm{op}_1, \ldots, \mathrm{op}_m}$ where $\mathrm{op}_1, \ldots, \mathrm{op}_m$ are "nice".

Main ingredient: Feferman-Vaught-Mostowski theorems

Given

$$A_1 \equiv B_1, \quad \dots, \quad A_n \equiv B_n$$

we need

$$\mathrm{op}(A_1, \dots, A_n) \equiv \mathrm{op}(B_1, \dots, B_n)$$

Example

For $A_1 \equiv_{\mathsf{FO}} B_1$ and $A_2 \equiv_{\mathsf{FO}} B_2$,

$$A_1 + A_2 \equiv_{\mathsf{FO}} B_1 + B_2 \quad \text{and} \quad A_1 \times A_2 \equiv_{\mathsf{FO}} B_1 \times B_2.$$

For $A_1 \equiv_{\mathsf{MSO}} B_1$ and $A_2 \equiv_{\mathsf{MSO}} B_2$,

$$A_1 + A_2 \equiv_{\mathsf{MSO}} B_1 + B_2 \quad \text{but} \quad A_1 \times A_2 \not\equiv_{\mathsf{MSO}} B_1 \times B_2.$$

Proofs by combining game strategies!

Baby steps with $+$

Given open pathwise-embeddings

$$F(A_1) \xleftarrow{f_1} R_1 \xrightarrow{g_1} F(B_1) \quad \text{and} \quad F(A_2) \xleftarrow{f_2} R_2 \xrightarrow{g_2} F(B_2)$$

We want

$$F(A_1 + A_2) \longleftarrow (??) \longrightarrow F(B_1 + B_2)$$

Baby steps with $+$ and $\widehat{+}$

- $+$ is a symmetric monoidal operation on $\mathcal{R}(\sigma)$
- $\mathbb{E}_k$ is an opmonoidal comonad w.r.t. $+$, we have a "law"

$$\kappa\colon \mathbb{E}_k(A+B) \to \mathbb{E}_k(A) + \mathbb{E}_k(B)$$

- Therefore, $+$ lifts to $\widehat{+}$ on $\mathrm{CoAlg}(\mathbb{E}_k)$ such that

$$\begin{array}{ccc} \mathrm{CoAlg}(\mathbb{E}_k) \times \mathrm{CoAlg}(\mathbb{E}_k) & \xrightarrow{\widehat{+}} & \mathrm{CoAlg}(\mathbb{E}_k) \\ {\scriptstyle F\times F}\uparrow & & \uparrow{\scriptstyle F} \\ \mathcal{R}(\sigma) \times \mathcal{R}(\sigma) & \xrightarrow{\quad +\quad} & \mathcal{R}(\sigma) \end{array}$$

- $\widehat{+}$ sends open pathwise-embeddings to open pathwise-embeddings!

Kleisli laws for general operations

Given $\mathrm{op}\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{B}$ and $\kappa\colon \mathbb{D} \circ \mathrm{op} \to \mathrm{op} \circ (\mathbb{C}_1 \times \cdots \times \mathbb{C}_n)$ such that

$$
\begin{array}{ccc}
\mathbb{D} \circ \mathrm{op} & \xrightarrow{\kappa} & \mathrm{op} \circ (\mathbb{C}_1 \times \cdots \times \mathbb{C}_n) \\
{\scriptstyle \varepsilon_{\mathrm{op}}}\downarrow & \swarrow {\scriptstyle \mathrm{op}(\varepsilon_1,\ldots,\varepsilon_n)} & \\
\mathrm{op} & &
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{D} \circ \mathrm{op} & & \xrightarrow{\kappa} & & \mathrm{op} \circ (\mathbb{C}_1 \times \cdots \times \mathbb{C}_n) \\
{\scriptstyle \delta_{\mathrm{op}}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{op}(\delta_1,\ldots,\delta_n)} \\
\mathbb{D}^2 \circ \mathrm{op} & \xrightarrow{\mathbb{D}\kappa} & \mathbb{D} \circ \mathrm{op} \circ (\mathbb{C}_1 \times \cdots \times \mathbb{C}_n) & \xrightarrow{\kappa} & \mathrm{op} \circ (\mathbb{C}_1^2 \times \cdots \times \mathbb{C}_n^2)
\end{array}
$$

If CoAlg($\mathbb{D}$) has equalisers then op lifts to coalgebras and

$$
\begin{array}{ccc}
\mathrm{CoAlg}(\mathbb{C}_1) \times \cdots \times \mathrm{CoAlg}(\mathbb{C}_n) & \xrightarrow{\widehat{\mathrm{op}}} & \mathrm{CoAlg}(\mathbb{D}) \\
{\scriptstyle F \times \cdots \times F}\uparrow & & \uparrow{\scriptstyle F} \\
\mathcal{A}_1 \times \cdots \times \mathcal{A}_n & \xrightarrow{\mathrm{op}} & \mathcal{B}
\end{array}
$$

Axioms for preservation of open pathwise-embeddings

(OPE1) $\mathrm{op}\colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{A}$ restricts to embeddings.

(OPE2) Any path embedding

$$e\colon \mathbb{P} \rightarrowtail \widehat{\mathrm{op}}(\mathbb{A}_1, \ldots, \mathbb{A}_n)$$

has a "minimal" decomposition

$$\mathbb{P} \xrightarrow{e_0} \widehat{\mathrm{op}}(\mathbb{P}_1, \ldots, \mathbb{P}_n) \xrightarrow{\widehat{\mathrm{op}}(e_1,\ldots,e_n)} \widehat{\mathrm{op}}(\mathbb{A}_1, \ldots, \mathbb{A}_n),$$

for some path embeddings $\mathbb{P}_i \rightarrowtail \mathbb{A}_i$, for $i = 1, \ldots, n$.

A meta Feferman-Vaught-Mostowski-theorem

Theorem

If op *has a Kleisli law* κ *which satisfies (OPE1) and (OPE2), then*

$$A_i \leftrightarrow_{\mathbb{C}_i} B_i, \quad \textit{for } i = 1, \ldots, n,$$

implies

$$\mathrm{op}(A_1, \ldots, A_n) \leftrightarrow_{\mathbb{D}} \mathrm{op}(B_1, \ldots, B_n).$$

A meta Courcelle theorem

Theorem

Fix a sentence φ and class $\mathscr{C} = \langle B_1, \ldots, B_n \rangle_{\mathrm{op}_1,\ldots,\mathrm{op}_m}$ (of augmented graphs) such that each op_i has a Kleisli law which satisfies (OPE1) and (OPE2) w.r.t. comonads $\mathbb{E}_k$.

Then, deciding $A \models \varphi$, for an arbitrary $A \in \mathscr{C}$, runs in "parsing time" $+ \mathcal{O}(n)$.

Caveat: We also need to account for the translation that adds the I relation

Summary

- Recovered Courcelle's theorem for bounded tree-width and clique-width.
- "Deterministic algorithm" to obtain new Courcelle's and FVM theorems:
  1. Check functoriality of op and find a Kleisli law κ.
  2. Understand the lifted operation $\widehat{\mathrm{op}}$.
  3. Check (OPE1) and (OPE2).
- MSO theorems via
  - a translation into the two-sorted setting, and
  - relative $\mathbb{E}_k$ over the two-sorted structures.
- Kleisli law $\implies$ new FVM-type theorems for $\leftrightarrows_{\mathbb{C}}$ and $\cong_{\mathsf{CoAlg}(\mathbb{C})}$
- Potentially many new FVM-type theorems with other comonads.