Finiteness species of structures

Yamcats 26

Zeinab GALAL

January 10th, 2022

Quantitative semantics of linear logic

$$A \Rightarrow B = {!A} \multimap B$$

intuitionistic implication — exponential modality — linear implication

Girard (1980's):

- Type A: vector space $\mathcal{R}^{[\![A]\!]}$
- Type $A \multimap B$: vector space $\mathcal{R}^{[\![A]\!] \times [\![B]\!]}$
- Linear program $P : A \multimap B$: a matrix $[\![P]\!] \in \mathcal{R}^{[\![A]\!] \times [\![B]\!]}$ or a linear map $[\![P]\!] : \mathcal{R}^{[\![A]\!]} \to \mathcal{R}^{[\![B]\!]}$
- Interaction: composition of matrices

Controlling non-determinism

- Type $A \Rightarrow B$: vector space $\mathcal{R}^{\mathcal{M}_{fin}(\llbracket A \rrbracket) \times \llbracket B \rrbracket}$
- Program $P : A \Rightarrow B$: analytic map $\llbracket P \rrbracket : \mathcal{R}^{\llbracket A \rrbracket} \to \mathcal{R}^{\llbracket B \rrbracket}$ given by a power series

$$(\llbracket P \rrbracket(x))_b = \sum_{m \in \mathcal{M}_{fin}(\llbracket A \rrbracket)} \llbracket P \rrbracket_{(m,b)} \cdot x^m$$

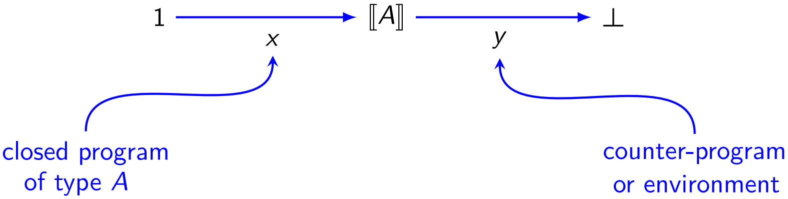
Issues

- Functional types lead to infinite dimensional vector spaces
- The sums need not to converge

Finiteness spaces are a way to control non-determinism by enforcing finite interactions.

Orthogonality on models of computation

C: fixed model of linear logic with monoidal units $1, \perp$.



- Orthogonality relation:

$$\perp_A \quad \subseteq \quad \mathbf{C}(1, [\![A]\!]) \times \mathbf{C}([\![A]\!], \perp)$$

- Well studied for 1-categories (Hyland-Schalk): allows for more control on interactions between programs and environments.

Relational model reminder

Definition

The *category of relations* (denoted **Rel**) is defined by

- objects: sets $A, B, \ldots$
- morphisms: binary relations $\mathbf{Rel}(A, B) := \mathcal{P}(A \times B)$
- identity: $id_A := \{(a, a) \mid a \in A\} \in \mathbf{Rel}(A, A)$
- composition: for $\mathcal{R} \in \mathbf{Rel}(A, B)$ and $\mathcal{S} \in \mathbf{Rel}(B, C)$, $\mathcal{S} \circ \mathcal{R} \subseteq A \times C$ is defined by:

$$\mathcal{S} \circ \mathcal{R} := \{(a, c) \mid \exists b \in B, (a, b) \in \mathcal{R} \text{ and } (b, c) \in \mathcal{S}\}$$

or equivalently $(\mathcal{S} \circ \mathcal{R})_{(a,c)} = \bigvee_{b \in B} \mathcal{R}_{(a,b)} \wedge \mathcal{S}_{(b,c)}$

Orthogonality on the Relational Model

Let $x \in \mathbf{Rel}(1, A) \cong \mathcal{P}(A)$ and $y \in \mathbf{Rel}(A, 1) \cong \mathcal{P}(A)$:

- Coherence spaces (Girard):

$$x \perp_A y \quad :\Leftrightarrow \quad |x \cap y| \leq 1$$

  stable semantics (minimal part of the input needed to compute a given output)

- Totality spaces (Loader):

$$x \perp_A y \quad :\Leftrightarrow \quad |x \cap y| = 1$$

  every computation yields exactly one result

Relational Finiteness Spaces

Ehrhard: for $x \in \mathbf{Rel}(1, A) \cong \mathcal{P}(A)$ and $y \in \mathbf{Rel}(A, 1) \cong \mathcal{P}(A)$,

$$x \perp y \quad :\Leftrightarrow \quad x \cap y \text{ is finite}$$

For a countable set A and $\mathcal{F} \subseteq \mathcal{P}(A)$, define

$$\mathcal{F}^{\perp} := \{y \in \mathcal{P}(A) \mid \forall x \in \mathcal{F}, x \perp y\}$$

$\rightsquigarrow$ Galois connection on the poset $\mathcal{P}(\mathcal{P}(A))$ ordered by inclusion:

$$\mathcal{P}(\mathcal{P}(A)) \underset{(-)^{\perp}}{\overset{(-)^{\perp}}{\underset{\perp}{\rightleftarrows}}} \mathcal{P}(\mathcal{P}(A))$$

Relational Finiteness Spaces

Definition (Ehrhard)

A relational *finiteness space* is a pair $(A, \mathcal{F}(A))$ where A is a countable set and $\mathcal{F}(A)$ is a subset of $\mathcal{P}(A)$ verifying $\mathcal{F}(A) = \mathcal{F}(A)^{\perp\perp}$.

Elements of $\mathcal{F}(A)$ are called *finitary subsets* as they "behave" like finite subsets:

- closure under inclusion
- closure under finite unions

$\mathcal{P}_{fin}(A)$	$\subseteq$	$\mathcal{F}(A)$	$\subseteq$	$\mathcal{P}(A)$
smallest finiteness structure				largest finiteness structure

Relational Finiteness Spaces

Definition

The category **FinRel** has objects finiteness spaces and morphisms are relations that preserve the finiteness structure.

For finiteness spaces $(A, \mathcal{F}(A))$ and $(B, \mathcal{F}(B))$, a relation $R \subseteq A \times B$ is in **FinRel** if it verifies:

- forward preservation: for all $x \in \mathcal{F}(A)$, $R_\star \cdot x \in \mathcal{F}(B)$
- backward preservation: for all $y \in \mathcal{F}(B)^\perp$, $R^\star \cdot y \in \mathcal{F}(A)^\perp$

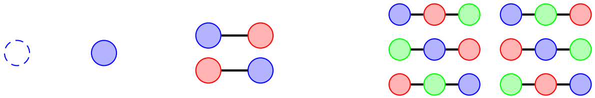
$$
\begin{array}{ccc}
\mathcal{P}(A) & \xrightarrow{R_\star} & \mathcal{P}(B) \\
\uparrow & & \uparrow \\
\mathcal{F}(A) & \dashrightarrow & \mathcal{F}(B)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(B) & \xrightarrow{R^\star} & \mathcal{P}(A) \\
\uparrow & & \uparrow \\
\mathcal{F}(B)^\perp & \dashrightarrow & \mathcal{F}(A)^\perp
\end{array}
$$

We obtain a model of finite non-determinism with iteration but no fixpoint.

Combinatorial Species Reminder

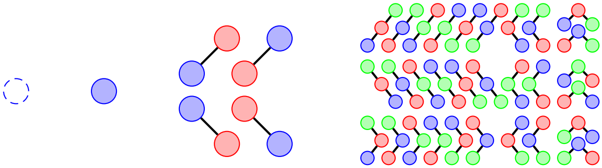
Count the number of ways combinatorial structures can be formed

- Lists or linear orderings



$$1 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + 2 \cdot \frac{x^2}{2!} + 6 \cdot \frac{x^3}{3!} + \dots = \sum_{n=0}^{+\infty} x^n$$

- Binary rooted trees



$$1 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + 4 \cdot \frac{x^2}{2!} + 30 \cdot \frac{x^3}{3!} + \dots = \sum_{n=0}^{+\infty} C_n \cdot x^n$$

Combinatorial Species Reminder

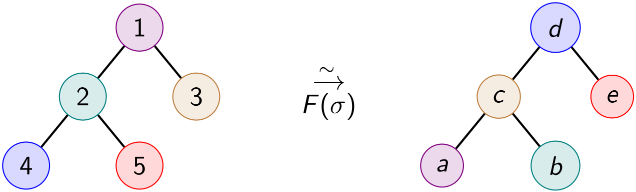
category $\mathbb{B}$ objects: finite sets, morphisms: bijections

Definition (Joyal 1981)

A *species of structure* is a functor $F : \mathbb{B} \to \mathbf{Set}$.

- Given a finite set of *labels* $\mathcal{U} \in \mathbb{B}$, an element $x \in F[\mathcal{U}]$ is called a *F-structure on $\mathcal{U}$*
- Given a bijection $\sigma : \mathcal{U} \xrightarrow{\sim} \mathcal{V} \in \mathbb{B}$, the bijection $F[\sigma] : F[\mathcal{U}] \xrightarrow{\sim} F[\mathcal{V}]$ is called the *transport of F-structures along σ*

$$\mathcal{U} = \{1, 2, 3, 4, 5\} \quad \underset{\sigma}{\xrightarrow{\sim}} \quad \mathcal{V} = \{a, b, c, d, e\}$$



Generalized species

- Fiore, Gambino, Hyland and Winskel 2008: generalized species as a model of differential linear logic

$$!\llbracket A \rrbracket \longrightarrow \widehat{\llbracket B \rrbracket}$$

free symmetric strict monoidal completion

presheaves over $\llbracket B \rrbracket$

- A $(\mathbf{1}, \mathbf{1})$-species of structure corresponds to a combinatorial species of structure

$$F : !\,\mathbf{1} \to \widehat{\mathbf{1}} \qquad \Leftrightarrow \qquad F : \mathbb{B} \to \mathbf{Set}$$

From Relations to Profunctors

$$R \subseteq A \times B \quad \Leftrightarrow \quad A \xrightarrow{\textit{function}} \mathcal{P}(B) \quad \Leftrightarrow \quad A \times B \xrightarrow{\textit{function}} 2$$

Definition

Let **A** and **B** be two categories, a *profunctor* from **A** to **B** is a functor

$$P : \mathbf{A} \to \widehat{\mathbf{B}} \qquad \text{(also denoted } P : \mathbf{A} \nrightarrow \mathbf{B}\text{)}$$

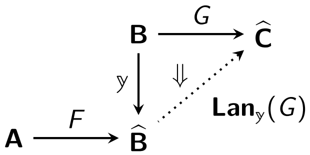
$$P : \mathbf{A} \nrightarrow \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \xrightarrow{\textit{functor}} \widehat{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} \times \mathbf{B}^{\mathrm{op}} \xrightarrow{\textit{functor}} \mathbf{Set}$$

Profunctor composition being not strictly associative, we need to work in the setting of bicategories

Bicategory of Profunctors

- **Objects:** small categories $\mathbf{A}, \mathbf{B}, \ldots$
- **1-cells:** profunctors $F : \mathbf{A} \nrightarrow \mathbf{B}$
- **2-cells:** natural transformations
- **Identity:** $\mathrm{id}_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$ is the Yoneda embedding $a \mapsto \mathbf{A}(-, a)$
- **Composition:** for $F : \mathbf{A} \nrightarrow \mathbf{B}$ and $G : \mathbf{B} \nrightarrow \mathbf{C}$,

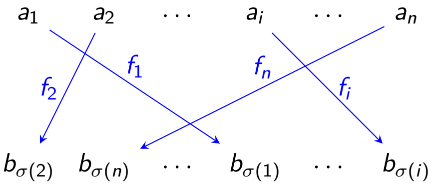
$$(G \circ F)(a, c) = \int^{b \in \mathbf{B}} F(a, b) \times G(b, c)$$



Linear Logic Structure

For a small category $\mathbf{A}$, define the category $!\mathbf{A}$:

- objects: finite sequences $\langle a_1, \ldots, a_n \rangle$ of objects of $\mathbf{A}$.
- morphisms: pairs $(\sigma, (f_i)_{i \in \underline{n}}) : \langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_n \rangle$ of a permutation $\sigma \in \mathfrak{S}_n$ and a finite sequence of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbf{A}$.



Definition

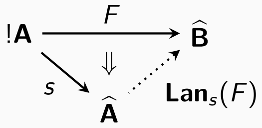
Given $\mathbf{A}$ and $\mathbf{B}$ two small categories, an $(\mathbf{A}, \mathbf{B})$-*generalized species of structures* is a profunctor $F : !\mathbf{A} \nrightarrow \mathbf{B}$.

Generalized species and analytic functors

What is the series counterpart of generalized species?

Definition (Fiore et al. 2008)

For small categories $\mathbf{A}$ and $\mathbf{B}$, a functor $P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ is *analytic* if there exists a generalized species $F : !\mathbf{A} \nrightarrow \mathbf{B}$ such that $P \cong \mathbf{Lan}_s F$



$$\text{where } s : \langle a_1, \ldots a_n \rangle \mapsto \sum_{i=1}^{n} y_{\mathbf{A}}(a_i).$$

- Example: $F : !\mathbf{A} \nrightarrow \mathbf{A}$ generalized species of binary trees

$$\mathbf{Lan}_s F : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$$
$$(X, a) \mapsto \sum_n C_n \times X^n(a)$$

Setting for Finiteness Species

We work with locally finite categories (homsets are finite)

- The yoneda embedding is valued in finite presheaves i.e. $\mathbb{y}_{\mathbf{A}} : \mathbf{A} \to [\mathbf{A}^{op}, \mathbf{FinSet}]$.
- Finitely presentable presheaves $[\mathbf{A}^{op}, \mathbf{Set}]_{fp}$ can be seen as a subcategory of finite presheaves $[\mathbf{A}^{op}, \mathbf{FinSet}]$.

Orthogonality

profunctors in $\mathbf{Prof}(\mathbf{1}, \mathbf{A}) \quad \Leftrightarrow \quad$ presheaves $X \in \widehat{\mathbf{A}}$

profunctors in $\mathbf{Prof}(\mathbf{A}, \mathbf{1}) \quad \Leftrightarrow \quad$ co-presheaves $Y \in \widehat{\mathbf{A}^{\mathrm{op}}}$

$$\mathbf{1} \xrightarrow[X]{\quad | \quad} \mathbf{A} \xrightarrow[Y]{\quad | \quad} \mathbf{1}$$

Orthogonality relation

For $X \in \mathbf{Prof}(\mathbf{1}, \mathbf{A})$ and $Y \in \mathbf{Prof}(\mathbf{A}, \mathbf{1})$,

$$X \perp Y \quad :\Leftrightarrow \quad Y \circ X = \int^{a \in \mathbf{A}} X(a) \times Y(a) \in \mathbf{FinSet}$$

Categorical finiteness structures

Given a subcategory $\mathcal{F} \hookrightarrow [\mathbf{A}^{op}, \mathbf{FinSet}]$, let $\mathcal{F}^{\perp}$ be the full subcategory of $[\mathbf{A}, \mathbf{FinSet}]$ whose object set is

$$\{Y : \mathbf{A} \to \mathbf{FinSet} \mid \forall X \in \mathcal{F}, X \perp Y\}$$

Definition

A *categorical finiteness structure* is a pair $(\mathbf{A}, \mathcal{F}(\mathbf{A}))$ of a locally finite category $\mathbf{A}$ and a full subcategory of $\mathcal{F}(\mathbf{A}) \hookrightarrow [\mathbf{A}^{op}, \mathbf{FinSet}]$ verifying $\mathcal{F}(\mathbf{A}) \cong (\mathcal{F}(\mathbf{A}))^{\perp\perp}$.

Finiteness presheaves

Objects of $\mathcal{F}(\mathbf{A})$ "behave" like finitely presentable objects:

- closure under retractions: if X' is a retract of $X \in \mathcal{F}(\mathbf{A})$, then $X' \in \mathcal{F}(\mathbf{A})$.
- closure under finite colimits: a finite colimit of elements of $\mathcal{F}(\mathbf{A})$ is in $\mathcal{F}(\mathbf{A})$.

$[\mathbf{A}^{op}, \mathbf{Set}]_{fp}$	$\hookrightarrow$	$\mathcal{F}(\mathbf{A})$	$\hookrightarrow$	$[\mathbf{A}^{op}, \mathbf{FinSet}]$
smallest finiteness structure				largest finiteness structure

Finiteness Profuntors

Definition

For finiteness structures $(\mathbf{A}, \mathcal{F}(\mathbf{A}))$, $(\mathbf{B}, \mathcal{F}(\mathbf{B}))$, $P : \mathbf{A} \times \mathbf{B}^{op} \to \mathbf{FinSet}$ is *a finiteness profunctor* if $\mathbf{Lan}_{\mathbb{y}_{\mathbf{A}}}(P)$ can be factored as follows:

$$\begin{array}{ccc} \widehat{\mathbf{A}} & \xrightarrow{\mathbf{Lan}_{\mathbb{y}_{\mathbf{A}}}(P)} & \widehat{\mathbf{B}} \\ \uparrow & & \uparrow \\ \mathcal{F}(\mathbf{A}) & \dashrightarrow & \mathcal{F}(\mathbf{B}) \end{array}$$

Denote by **FinProf** the bicategory of finiteness structures, finiteness profunctors and natural transformations.

FinProf as a Model of Linear Logic

Definition

For a finiteness structure $(\mathbf{A}, \mathcal{F}\mathbf{A})$, we define $!(\mathbf{A}, \mathcal{F}(\mathbf{A})) := (!\mathbf{A}, \mathcal{F}(!\mathbf{A}))$ where $\mathcal{F}!\mathbf{A} := \{X^! \mid X \in \mathcal{F}\mathbf{A}\}^{\perp\perp}$.

For a finite presheaf $X : \mathbf{A}^{op} \to \mathbf{FinSet}$, its lifting $X^! : (!\mathbf{A})^{op} \to \mathbf{FinSet}$ is given by

$$X^! : \langle a_1, \ldots, a_n \rangle \in !\mathbf{A} \mapsto \prod_{i=1}^{n} X(a_i)$$

is also a finite presheaf.

Theorem

The co-Kleisli bicategory $\mathbf{FinProf}_!$ *is cartesian closed.*

Examples

- Morphisms $!(\mathbf{1}, \mathcal{F}\mathbf{1}) \twoheadrightarrow (\mathbf{1}, \mathcal{F}(\mathbf{1}))$ are species whose analytic functor is polynomial:
  - Analytic functor with finite support **Set** $\to$ **Set**

$$P : X \mapsto \mathbf{1} + X + X^2/\mathfrak{S}_2 + \cdots + X^n/\mathfrak{S}_n$$

  - Analytic functor (but not in the finiteness model) **Set** $\to$ **Set**

$$L : X \mapsto \mathbf{1} + X + X^2 + \cdots + X^n + \ldots \cong \sum_{n \in \mathbb{N}} X^n$$

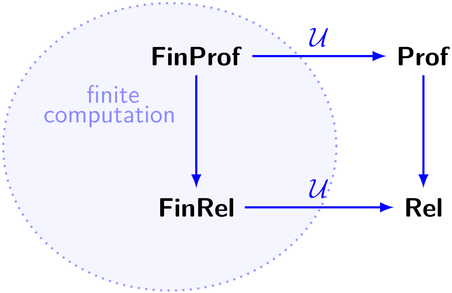
  $\Rightarrow$ no fixpoints ($L \cong \mathbf{1} + X \cdot L$)

- In higher types, we can have infinite support:

$$!(!(\mathbf{1}, \mathcal{F}\mathbf{1}) \multimap (\mathbf{1}, \mathcal{F}\mathbf{1})) \twoheadrightarrow \mathbf{1}$$

$$\text{“}E \mapsto E(0)\text{”}$$

Conclusion and next steps



- Categorify the orthogonality construction
- Generalize to enriched species (in particular for species enriched over vector spaces to remain in a finite dimensional setting).
- Replace finite presentability by other classes of objects

Thank you for your attention