∞-connected objects

1-topos — A category which "behaves" like the category of Sets.

- Has internal language

  $\Rightarrow, \vee, \wedge, \exists, \forall \qquad \{x \mid p\}$
- Has a Ω subobject classifier

  "set of truth values"
- locally cart closed.

∞-topos — An ∞-category which "behave" like the ∞-category of ∞-groupoids

Set $\qquad$ ∞-groupoid $\rightsquigarrow$ "proof relevant set"

- Has an internal language
- $\mathbb{N}$, Bool, $S^n \leftarrow$ spheres
- $\mathcal{U}$ - "an type of types"

  $\rightsquigarrow \Omega$ propositions definable

A type is a collection structure

A prop is a type in which this structure is trivial

- Logic of proof relevant equality which we interpret geometric

Propositional Truncation

$$\begin{tikzcd} \Omega \arrow[r, hook] & \mathcal{U} \arrow[l, bend right, "|-|_{-1}"'] \end{tikzcd}$$

$\to$ $x : X$ $\rightsquigarrow$ "I know a element of X"

$(1 \to X)$

$x' : |X|_{-1}$ $\rightsquigarrow$ "I know there is some element of x"
(but not which one)

Connectivity

compare:

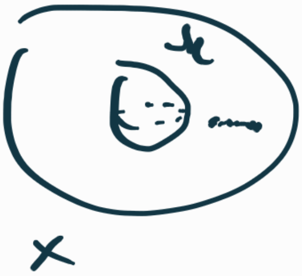
$$\text{is-prop} := \prod_{x\,y} x = y$$

$$\text{is-conn} := \prod_{x\,y : X} |x = y|_{-1}$$

Def A space is n-conn if for every $k \leq n$

$$X \twoheadrightarrow X^{S^k}$$

Eg.



For every map $\phi : S^k \to X$

$$\begin{tikzcd} S^k \arrow[r] \arrow[d, dotted] & X \\ D^k \arrow[ur, dashed] & \end{tikzcd}$$

T.F.A.E

1. X is n-conn
2. $\to$ $\pi_k(X) = 0$ $\quad k \leq n$
3. L.L.P. w.r.t. all n-truncated maps.

Obs: This makes sense for $n = \infty$

We say such objects are ∞-connected.

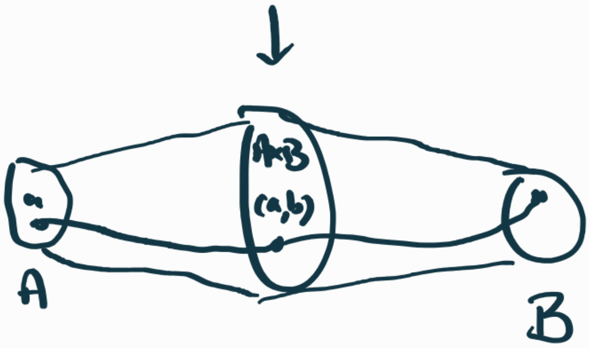
Classically Whitehead's theorem (1949)

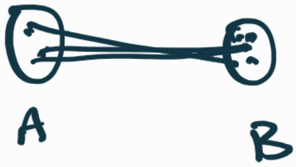
Any ∞-connected is contractible.

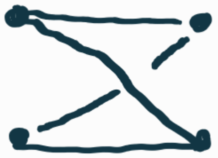
Join

$$\begin{tikzcd} A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "i_2"] \\ A \arrow[r, "i_1"'] & A * B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

(homotopy pushout!)







$S^0 * S^0$

S^1

$$S^0 * A = \Sigma A$$

$$\underbrace{S^0 * \cdots\cdots * S^0}_{n+1} = S^n$$

Thm If A is n-conn
B is m-conn
$A * B$ is $(n+m+2)$-conn

Thm $A \to A^{*2} \to A^{*3} \to \cdots \to A^{*\infty} \simeq |A|_{-1}$

$a : A \qquad y : A^{*\infty}$

$y : A^{*n} * A$

Construction I

$$S^0 \to S^1 \to S^2 \to \cdots\cdots \to S^\infty = |S^0|_{-1} \simeq *$$

Def X is a "ghost S^0" if

$$|X \simeq S^0|_{-1} \quad (\mathbb{Z}/2\text{-torsor})$$

Given $\{X_i\}$ where $|X_i \simeq S^0|_{-1}$ ← all X_i are (-1)-conn

$$\text{I:} \quad \underset{0}{X_0} \hookrightarrow \underset{1}{X_0 * X_1} \hookrightarrow \underset{2}{X_0 * X_1 * X_2} \to \cdots \underset{\infty}{\mathop{\ast}_{k=0}^{\infty} X_k} = X_\infty$$

$-1 + -1 + 2$

Claim X_∞ is ∞-conn.

but it is not provable the X_∞ is contr.

Construction

Given P_i Q_i $0 \le i \le \infty$ P_i, Q_i props

$$P_i * Q_i = |P_i \sqcup Q_i|_{-1} \simeq 1$$

Let $X_i = P_i \sqcup Q_i$ each X_i is (-1)-conn

$$\underset{(-1)}{X_0} \hookrightarrow \underset{0}{X_0 * X_1} \hookrightarrow \underset{1}{X_0 * X_1 * X_2} \hookrightarrow \cdots \to X_\infty$$

$\pi_n(X)$

pick a rep $S^n \to X$ for every element of $\pi_n(X)$

$X \to |X|_0$

$X \longrightarrow X^{\circ}$

$$\prod_{x,y:X} |x=y|_{-1}$$

$$\frac{\prod_{x,y:X} \prod_{p,q:x=y} |p=q|_{-1}}{S^1 \xrightarrow{f} X}$$

$$\begin{cases} S^0 & (-1)\text{-con} \\ (-1)+(-1)+2=0 \end{cases}$$

1-conn

$$\boxed{8^{S^1}}$$