Some Constructions of Split Comprehension Categories with Application to Realizability

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July 29, 2016

We are working with *Grothendieck fibrations* and *comprehension categories*. Suppose we are given a functor $p : \mathbb{E} \to \mathbb{B}$.

Definition

A morphism $f : X \to Y$ in $\mathbb{E}$ is *cartesian over* $u : I \to J$ in $\mathbb{B}$ if $p(f) = u$ and for every $g : Z \to Y$ for which there exists w with $p(g) = u \circ w$, there is a unique $h : Z \to X$ in $\mathbb{E}$ such that $p(h) = w$ and $f \circ h = g$. In a diagram:

$$
\begin{tikzcd}
 & & Z \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "g"] & & \\
\mathbb{E} \arrow[ddd, "p"'] & & & X \arrow[r, "f"'] & Y \\
 & & & & \\
 & & pZ \arrow[dr, "w"'] \arrow[drr, bend left, "u \circ w = pg"] & & \\
\mathbb{B} & & & I \arrow[r, "u"] & J
\end{tikzcd}
$$

For example, let $\mathbb{B}$ be any category, let $\mathbb{E}$ be $\mathbb{B}^{\to}$, the category of arrows over $\mathbb{B}$, and let p be the codomain functor. Then cartesian maps in $\mathbb{B}^{\to}$ are precisely pullbacks:

$$\begin{tikzcd}
Z_1 \arrow[d] \arrow[dr, dotted, "h"'] \arrow[drr, bend left, "g"] & & \\
Z_0 \arrow[dr, "w"'] & X_1 \arrow[r, "f"'] \arrow[d] & Y_1 \arrow[d] \\
& X_0 \arrow[r, "u"] & Y_0
\end{tikzcd}$$

We think of objects of $\mathbb{B}$ as "contexts", objects of $\mathbb{E}$ as "dependent types", and cartesian maps as "substitutions into types."

Definition

1. A functor $p : \mathbb{E} \to \mathbb{B}$ is a *(Grothendieck) fibration* if for every $u : I \to J$ in $\mathbb{B}$ and object Y in $\mathbb{E}$ with $p(Y) = J$, there exists an object X of $\mathbb{E}$ and a cartesian morphism $f : X \to Y$ such that $p(f) = u$.
2. A *cloven fibration* is a functor $p : \mathbb{E} \to \mathbb{B}$ together with a choice of cartesian morphisms: for every $u : I \to J$ and Y with $p(Y) = J$, an object $u^*(Y)$ of $\mathbb{E}$, and a cartesian morphism $\overline{u}(Y)$ (we refer to the extra data as a *cleavage*.
3. A *split fibration* is a cloven fibration, such that the cleavage respects identities and composition:
   3.1 For all I, $\mathrm{id}_I^*(Y) = Y$ and $\overline{\mathrm{id}_I} = \mathrm{id}_Y$.
   3.2 For all $u : I \to J$ and $v : J \to K$ and Y with $p(Y) = K$, $(v \circ u)^*(Y) = u^*(v^* Y)$ and $\overline{v \circ u} = \overline{v} \circ \overline{u}$.

Eg. $\mathrm{cod} : \mathbb{B}^{\to} \to \mathbb{B}$ is a fibration precisely if $\mathbb{B}$ has pullbacks.
We refer to $\mathbb{B}$ as the *base category* and $\mathbb{E}$ as the *total category*.

Example

Let $\mathbb{B} := \mathrm{Set}$. Let $\mathbb{E} := \mathrm{Fam}(\mathrm{Set})$ be the category of families of sets. That is, objects of Fam(Set) consist of a set I together with a family of sets $(X_i)_{i \in I}$. A morphism from $(X_i)_{i \in I}$ to $(Y_j)_{j \in J}$ is a function $u : I \to J$, together with functions $X_i \to Y_{u(i)}$ for each $i \in I$.

The projection functor $\mathrm{Fam}(\mathrm{Set}) \to \mathrm{Set}$ is equivalent to the codomain functor $\mathrm{Set}^{\to} \to \mathrm{Set}$.

There is a canonical splitting on Fam(Set) defined as follows. For $u : I \to J$, $u^*((Y_j)_{j \in J})$ is the family indexed by I defined below:

$$(u^*((Y_j)_{j \in J}))_i := Y_{u(i)}$$

Definition

An *assembly* consists of a set X together with a function $\alpha : X \to \mathcal{P}_i(\mathbb{N})$ (where $\mathcal{P}_i(\mathbb{N})$ denotes inhabited subsets of $\mathbb{N}$).

If (X, α) and (Y, β) are assemblies and $g : X \to Y$ is a function and $a \in \mathbb{N}$, we say f is *tracked* by a if for all $x \in X$, and all $b \in \alpha(x)$, $\phi_a(b)$ is defined and belongs to $\beta(g(x))$.

Assemblies form a category Asm, where morphisms $(X, \alpha) \to (Y, \beta)$ are functions $f : X \to Y$ such that there exists $a \in \mathbb{N}$ that tracks f.

A *uniform family* consists of an assembly (I, β), together with a family of assemblies $(X_i, \alpha_i)_{i \in I}$. Uniform families form a category UFam(Asm).

Projection UFam(Asm) $\to$ Asm is a split fibration, which is equivalent to the codomain fibration Asm$^{\to}$ $\to$ Asm (analogously to Fam(Set) $\to$ Set).

Definition (Jacobs)

A *comprehension category* consists of categories $\mathbb{B}$ and $\mathbb{E}$ together with functors p and χ in the following commutative diagram, such that p is a fibration and χ preserves cartesian maps.

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathbb{B}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathbb{B} &
\end{tikzcd}$$

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Basic idea: Objects I of $\mathbb{B}$ are contexts. The fibre $p^{-1}(I)$ is dependent types in context I. If I is an object of $\mathbb{B}$ and $X \in p^{-1}(I)$, we think of $\mathrm{dom}(\chi(X))$ as the new context resulting from extending I with the dependent type X. Terms of type X are sections of $\chi(X)$. The splitting tells us how to substitute into types.

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- The same ideas should apply to both cubical sets and cubical assemblies.
- Simple enough for me to understand.

Given a split fibration, we can produce a new split fibration via the following observation.

Definition

Let $r : \mathbb{F} \to \mathbb{E}$ and $p : \mathbb{E} \to \mathbb{B}$. We say r *lifts cartesian maps uniquely* if for object Z of $\mathbb{F}$ and every map $f : X \to r(Z)$ in $\mathbb{E}$ that is cartesian over p, there is a unique object X' of $\mathbb{F}$ and morphism $g' : X' \to Z$ such that g is cartesian over $p \circ r$.

Lemma

Suppose that we are given a splitting on p and r lifts cartesian maps uniquely. Then the splitting on p lifts uniquely to a splitting on $p \circ r$.

Using a key idea from the van den Berg-Garner interpretation of type theory, we get the following.

Lemma

1. *Let* $p : \mathbb{E} \to \mathbb{B}$ *be a functor, and let* $T : \mathbb{E} \to \mathbb{E}$ *be an endofunctor over* p. *Let* T-Alg *be the category of* T-*algebras, and let* $r : T$-Alg $\to \mathbb{E}$ *be the forgetful functor. Then* r *lifts cartesian maps uniquely.*
2. *If* η *is a unit making* (T, η) *a pointed endofunctor over* p, *then the forgetful functor* $r : (T, \eta)$-Alg $\to \mathbb{E}$ *lifts cartesian maps uniquely.*
3. *If* μ *is a unit making* (T, η, μ) *a monad over* p, *then the forgetful functor* $r : (T, \eta, \mu)$-Alg $\to \mathbb{E}$ *lifts cartesian maps uniquely.*

The same is true for coalgebras if we add the assumption that T preserves cartesian maps.

Let $\mathbb{B}$ be a category. Every awfs gives us a monad R on $\mathbb{B}^{\to}$. If we are given a split CC and an awfs on the same category, $\mathbb{B}$, we combine them via pullback:

$$\begin{tikzcd}
\mathbb{E} \times_{\mathbb{B}^{\to}} R\text{-Alg} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & R\text{-Alg} \arrow[d] \\
\mathbb{E} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathbb{B}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathbb{B} &
\end{tikzcd}$$

The property of lifting cartesian maps uniquely is preserved by pullback along cartesian maps. So using the splitting on p, we obtain a splitting on the composition $\mathbb{E} \times_{\mathbb{B}^{\to}} R\text{-Alg} \to \mathbb{B}$. This gives us a new split CC with total category $\mathbb{E} \times_{\mathbb{B}^{\to}} R\text{-Alg}$.

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Applying this to Hofmann's interpretation of extensional type theory in cubical sets and suitable awfs's gives us precisely the CwFs defined by Coquand et al.

Split CC for extensional type theory + awfs

$$\downarrow$$

Split CC for intensional type theory

- For defining the split CC we don't need all the structure of an awfs. For this and some other things the weaker notion of left algebraic weak factorisation system (lawfs) suffices.
- We can use strong coproducts in $\mathbb{E}$ and the structure of an lawfs to implement Σ-types
- Under certain conditions Π-types and Id-types can also be added.

Before we had an endofunctor on the total category. We can also work with an endofunctor on the base category. This is done using two pullbacks.

$$
\begin{tikzcd}
G^*(\mathbb{E}) \arrow[rr, "G^*(\chi)"] \arrow[dr] \arrow[drr] & & G^*(\mathbb{B}^{\to}) \arrow[dl] \arrow[drr] & & \\
& T\text{-Alg} \arrow[drr, "G"'] & \mathbb{E} \arrow[rr, "\chi"] \arrow[dr] & & \mathbb{B}^{\to} \arrow[dl] \\
& & & \mathbb{B} &
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G^*(\mathbb{E}) \arrow[rr, "G^*(\chi)"] \arrow[dr] \arrow[drr] & & G^*(\mathbb{B}^{\rightarrow}) \arrow[dl] \arrow[drr] & & \\
& T\text{-Alg} \arrow[drr, "G"'] & \mathbb{E} \arrow[rr, "\chi"] \arrow[dr] & & \mathbb{B}^{\rightarrow} \arrow[dl] \\
& & & \mathbb{B} &
\end{tikzcd}
$$

First we base change along the forgetful functor $G : T\text{-Alg} \to \mathbb{B}$. It is well known that splitness is preserved by base changes. Then we pullback along the functor $T\text{-Alg}^{\rightarrow} \to G^*(\mathbb{B}^{\rightarrow})$, which lifts cartesian maps uniquely.

Definition

Let $\mathbb{C}$ be a category. An *internal category* consists of

1. an object C_0 ("the object of objects")
2. an object C_1 ("the object of morphisms")
3. morphisms $\partial_0, \partial_1 : C_1 \to C_0$ ("domain and codomain")
4. a morphism $i : C_0 \to C_1$ ("identity objects")
5. a morphism $m : C_1 \times_{C_0} C_1 \to C_1$ ("morphism of composition")
6. satisying certain axioms (commutative diagrams corresponding to associativity and identities)

Theorem (Bénabou and Roubaud, Beck)

Given a fibration $p : \mathbb{E} \to \mathbb{B}$ with dependent coproducts and an internal category $\mathcal{C} := (C_0, C_1, \partial_0, \partial_1, i, m)$ in $\mathbb{B}$, one can construct a monad on $\mathbb{E}_{C_0}$ whose algebras are internal presheaves over $\mathcal{C}$.

We can recover a split CC corresponding to Hofmann's interpretation of extensional type theory in presheaves over a category $\mathcal{C}$ as follows.

1. View $\mathcal{C}$ as an internal category in Set.
2. $\mathrm{Fam}(\mathrm{Set})_{C_0}$ consists of families of sets indexed by C_0, which are just functions X from objects of $\mathcal{C}$ to Set.
3. Bénabou-Roubaud-Beck gives us a monad on $\mathrm{Fam}(\mathrm{Set})_{C_0}$. An algebra structure on $X : C_0 \to \mathrm{Set}$ is precisely the extra structure needed to make X into a functor $\mathcal{C} \to \mathrm{Set}$.
4. We have a split fibration with base category $\mathrm{Fam}(\mathrm{Set})_{C_0}$. Objects of the total category consist of pairs (X, F), with $X \in \mathrm{Fam}(\mathrm{Set})_{C_0}$ and $F \in \mathrm{Fam}(\mathrm{Set})_{\mathrm{dom}(\chi(X))}$.
5. Combining this split fibration with the monad gives us a new split fibration. The fibre of the total category over X is strictly isomorphic (not just equivalent) to the category $[\int X, \mathrm{Set}]$ (which is how dependent types are implemented in presheaves).

Split CC for extensional type theory (eg families of sets, uniform families of assemblies) $+$ internal category

$$\downarrow$$

New split CC for extensional type theory (eg cubical sets)

An internal category in the category of assemblies consists of:

1. An internal category $(C_0, C_1, \partial_0, \partial_1, i, m)$ in Set (ie a small category).
2. Existence predicates $E_0 : C_0 \to \mathcal{P}_i(\mathbb{N})$ and $E_1 : C_1 \to \mathcal{P}_i(\mathbb{N})$.
3. $\partial_0, \partial_1, i, m$ are tracked functions.

By defining suitable internal categories and applying the internal presheaf construction, we get

1. Stekelenburg's definition of simplicial assembly
2. A notion of cubical assembly (with optional diagonals and connections) corresponding to cubical sets
3. 01-substitution assemblies, corresponding to 01-substitution sets

Each of these is the base category of a split CC.

The awfs's on cubical sets and 01-substitution sets can be lifted to cubical assemblies and 01-substitution assemblies by defining an appropriate existence predicate.

If the existence predicate for the awfs (L, R) is defined appropriately, then for an algebra structure $Rf \to f$ to be tracked says precisely that one can compute the Kan filling operator.

(Recall that the algebra structures in cubical sets correspond precisely to Kan filling operators.)

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(Recall that the algebra structures in cubical sets correspond precisely to Kan filling operators.)

Unfolding the definitions easily yields an explicit description of a split CC.

Some miscellaneous remarks...

- I expect HoTT can be interpreted in cubical assemblies by adapting existing work (Coquand et al, Pitts and Orton).
- Cubical assemblies are not toposes or even pretoposes and are not complete, cocomplete or exact and have no subobject classifiers. This is fine.
- We cannot apply the usual version of Garner's small object argument, but an "internal" version might work.
- Instead of assemblies one can use PERs, modest sets or realizability toposes.
- Instead of constructing presheaves internally in assemblies, we can construct assemblies internally in presheaf categories. This might be useful for studying cubical type theory.