Principal equivalence relations

Egbert Rijke

Carnegie Mellon University
erijke@andrew.cmu.edu

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Part I: Classifying types (jww Ulrik Buchholtz)

Overview

- Definition of $\mathbb{R}\mathrm{P}^n$.
- Definition of $\mathbb{C}\mathrm{P}^n$.
- Principal H-spaces ('infinitely coherent groups').
- The general projective space construction.
- The principal H-space structure of loop spaces.
- Principal H-spaces are pointed connected types.

$$\mathbb{R}\mathsf{P} : \mathbb{N}_{-1} \to \mathsf{U}$$
$$\mathsf{cov}_2 : \prod_{(n:\mathbb{N}_{-1})} \mathbb{R}\mathsf{P}^n \to \mathsf{U}_{\mathbf{2}}$$

$$\begin{tikzcd}
\sum_{(x:\mathbb{R}\mathsf{P}^n)} \mathsf{cov}_2^n(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{R}\mathsf{P}^n \arrow[r] \arrow[drr, bend right, "\mathsf{cov}_2^n"'] & \mathbb{R}\mathsf{P}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\mathsf{cov}_2^{n+1}"'] & \\
 & & \mathsf{U}_{\mathbf{2}}
\end{tikzcd}$$

- For each $n : \mathbb{N}$ there is an equivalence

$$\mathbb{S}^n \simeq \sum_{(x:\mathbb{R}\mathsf{P}^n)} \mathsf{cov}_2^n(x).$$

- We obtain the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathbb{R}\mathsf{P}^n) \longrightarrow \pi_k(\mathbf{2}) \longrightarrow \pi_k(\mathbb{S}^n) \longrightarrow \pi_k(\mathbb{R}\mathsf{P}^n) \longrightarrow \cdots$$

  Since $\pi_k(\mathbf{2}) = 0$ for $k \geq 1$, we get the isomorphisms

$$\pi_k(\mathbb{S}^n) = \pi_k(\mathbb{R}\mathsf{P}^n)$$

  for $k \geq 2$.
- The map

$$\mathsf{cov}_2^\infty : \mathbb{R}\mathsf{P}^\infty \to \mathsf{U}_\mathbf{2}$$

  is an equivalence.

$$\mathcal{O}_{\mathbb{S}^1} : \mathsf{U}_{\mathbb{S}^1} \to \mathsf{U}$$

such that $\sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A) \times A$ is contractible

$$\begin{aligned}
\mathbb{C}\mathsf{P} &: \mathbb{N}_{-1} \to \mathsf{U} \\
\mathsf{cov}_{\mathbb{S}^1} &: \prod_{(n:\mathbb{N}_{-1})} \mathbb{C}\mathsf{P}^n \to \mathsf{U}_{\mathbb{S}^1} \\
\mathsf{orient}_{\mathbb{S}^1} &: \prod_{(n:\mathbb{N}_{-1})} \prod_{(x:\mathbb{C}\mathsf{P}^n)} \mathcal{O}_{\mathbb{S}^1}(\mathsf{cov}^n_{\mathbb{S}^1}(x))
\end{aligned}$$

$$\begin{tikzcd}
\sum_{(x:\mathbb{C}\mathsf{P}^n)} \mathsf{cov}^n_{\mathbb{S}^1}(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathbb{C}\mathsf{P}^n \arrow[r] \arrow[drr, bend right, "{\langle \mathsf{cov}^n_{\mathbb{S}^1}, \mathsf{orient}^n_{\mathbb{S}^1} \rangle}"'] & \mathbb{C}\mathsf{P}^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\mathsf{cov}^{n+1}_{\mathbb{S}^1}"'] & \\
& & \sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A)
\end{tikzcd}$$

- For each $n : \mathbb{N}$ there is an equivalence
$$\mathbb{S}^{2n+1} \simeq \sum_{(x:\mathbb{C}\mathsf{P}^n)} \mathsf{cov}^n_{\mathbb{S}^1}(x).$$
- We obtain the long exact sequence
$$\cdots \longrightarrow \pi_{k+1}(\mathbb{C}\mathsf{P}^n) \longrightarrow \pi_k(\mathbb{S}^1) \longrightarrow \pi_k(\mathbb{S}^{2n+1}) \longrightarrow \pi_k(\mathbb{C}\mathsf{P}^n) \longrightarrow \cdots$$
Since $\pi_k(\mathbb{S}^1) = 0$ for $k \geq 2$, we get the isomorphisms
$$\pi_k(\mathbb{S}^{2n+1}) = \pi_k(\mathbb{C}\mathsf{P}^n)$$
for $k \geq 3$.
- The map
$$\mathsf{cov}^\infty_{\mathbb{S}^1} : \mathbb{C}\mathsf{P}^\infty \to \sum_{(A:\mathsf{U}_{\mathbb{S}^1})} \mathcal{O}_{\mathbb{S}^1}(A)$$
is an equivalence.

Definition

A principal H-space structure on a type X with base point 1_X, consists of

1. a type family $\mathcal{O}_X : \mathsf{U}_X \to \mathsf{U}$ of orientations,
2. a canonical orientation $o_X : \mathcal{O}_X(X)$,

such that the type

$$\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$$

is contractible.

The classifying type of a principal H-space X is defined to be

$$\mathbf{B}X :\equiv \sum_{(A:\mathsf{U}_X)} \mathcal{O}_X(A).$$

This is a pointed connected type with loop space X.

The general projective space construction

$$
\begin{aligned}
\mathsf{P}(X) &: \mathbb{N}_{-1} \to \mathsf{U} \\
\mathsf{cov}_X &: \textstyle\prod_{(n:\mathbb{N}_{-1})} \mathsf{P}(X)^n \to \mathsf{U}_X \\
\mathsf{orient}_X &: \textstyle\prod_{(n:\mathbb{N}_{-1})} \prod_{(x:\mathsf{P}(X)^n)} \mathcal{O}_X(\mathsf{cov}_X^n(x))
\end{aligned}
$$

$$
\begin{tikzcd}
\sum_{(x:\mathsf{P}(X)^n)} \mathsf{cov}_X^n(x) \arrow[r] \arrow[d, "\mathsf{pr}_1"'] & \mathbf{1} \arrow[d] \arrow[ddr, bend left] & \\
\mathsf{P}(X)^n \arrow[r] \arrow[drr, bend right, "{\langle \mathsf{cov}_X^n, \mathsf{orient}_X^n \rangle}"'] & \mathsf{P}(X)^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "{\langle \mathsf{cov}_X^{n+1}, \mathsf{orient}_X^{n+1} \rangle}"'] & \\
 & & \sum_{(A:\mathsf{U}_X)} \mathcal{O}_X(A)
\end{tikzcd}
$$

- For each $n : \mathbb{N}$ there is an equivalence

$$X^{*(n+1)} \simeq \sum_{(x:\mathsf{P}(X)^n)} \mathsf{cov}_X^n(x).$$

- We obtain the long exact sequence

$$\cdots \longrightarrow \pi_{k+1}(\mathsf{P}(X)^n) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X^{*(n+1)}) \longrightarrow \pi_k(\mathsf{P}(X)^n) \longrightarrow \cdots$$

- The map

$$\mathsf{cov}_X^\infty : \mathsf{P}(X)^\infty \to \mathbf{B}X$$

is an equivalence.

Lemma

Any loop space can be given the structure of a principal H-space.

Proof (Definition of the type of orientations).

Define $P : X_{x_0} \to \mathrm{Type}$ by $P(x) :\equiv (x = x_0)$. Then we have

$$\begin{tikzcd} \Omega(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[r] \arrow[d, "x_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U}_X \arrow[d] \\ \mathbf{1} \arrow[r, "x_0"'] & X_{x_0} \arrow[r, "P"'] & \mathrm{U}_X \end{tikzcd}$$

Now take

$$\mathcal{O}_X(A) :\equiv \mathrm{fib}_P(A)$$
$$o_X(\Omega(X)) :\equiv \langle x_0, \mathrm{refl}_{\Omega(X)} \rangle$$

□

Proof of the contractibility property.

We have a commuting triangle

$$\begin{tikzcd} X_{x_0} \arrow[rr, "\simeq"] \arrow[dr, "P"'] & & \sum_{(A:U_X)} \mathrm{fib}_P(A) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{U}_X & \end{tikzcd}$$

Therefore, the following are equivalent

- $\sum_{(A:U_X)} \mathcal{O}_X(A) \times A$ is contracitble
- $\sum_{(x:X_{x_0})} P(x)$ is contractible

The latter is obvious. $\square$

Theorem

The map **B** *from principal H-spaces to the type of (small) pointed connected types is an equivalence.*

Part II: Principal equivalence relations

Overview

- Principal equivalence relations (‘infinitely coherent type-valued equivalence relations’).
- The quotient approximation construction.
- The (modified) join construction.

Definition

A principal equivalence relation on a type A consists of

- A binary relation $R : A \to (A \to \mathsf{U})$ with a proof $\rho : \prod_{(a:A)} R(a,a)$ of reflexivity,
- A type family

$$\mathcal{O}_R : \mathrm{im}(R) \to \mathsf{U}$$

  of R-orientations on the predicates $P : A \to \mathsf{U}$ in the image of R, with a canonical R-orientation

$$o_R : \prod_{(a:A)} \mathcal{O}_R(R(a)),$$

such that the type

$$\sum_{(P:\mathrm{im}(R))} \mathcal{O}_R(P) \times P(a)$$

is contractible for every $a : A$.

- A principal equivalence relation on $\mathbf{1}$ is the same thing as a principal H-space.

Example

Let A be a type, and let $\bigcirc$ be a modality. Define

$$R(a,b) :\equiv \bigcirc(a=b)$$

Then the type

$$\textstyle\sum_{(P:\mathrm{im}(R))} P(a)$$

is contractible for any $a : A$. So we may take $\mathcal{O}_R(P) :\equiv \mathbf{1}$.

Definition

Given a principal equivalence relation R on A, we define the quotient

$$A/R :\equiv \sum_{(P:\mathrm{im}(R))} \mathcal{O}_R(P).$$

We define the quotient map $q_R : A \to A/R$ by

$$q_R(a) :\equiv \langle q'_R(a), o_R(a) \rangle.$$

Lemma

Given a principal equivalence relation R on A, we have by the encode-decode method an equivalence

$$(q_R(a) =_{A/R} q_R(b)) \quad \simeq \quad R(a,b)$$

for any $a, b : A$.

$$\begin{tikzcd}
\sum_{(a:A)} \sum_{(x:X)} Y(x,a) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d] \arrow[ddr, bend left, "\langle Y, o_Y \rangle"] & \\
A \arrow[r] \arrow[drr, bend right] & A +_Y X \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\langle Y', o_{Y'} \rangle"'] & \\
& & \sum_{(P:\mathrm{im}(R))} \mathcal{O}_R(P)
\end{tikzcd}$$

This defines an endomorphism on the type

$$\sum_{(X:\mathsf{U})} \sum_{(Y:X\to\mathrm{im}(R))} \prod_{(x:X)} \mathcal{O}_R(Y(x)).$$

Via this endomorphism, we define the quotient approximation sequence

$$\begin{tikzcd}
{[A/R]_0} \arrow[r, "\mathsf{inr}"] \arrow[drr, "{[q_R]_0}"'] & {[A/R]_1} \arrow[r, "\mathsf{inr}"] \arrow[dr, "{[q_R]_1}"] & {[A/R]_2} \arrow[r, "\mathsf{inr}"] \arrow[d, "{[q_R]_2}"'] & \cdots \arrow[dl, "{[q_R]_3}"] \\
& & \sum_{(P:\mathsf{im}(R))} \mathcal{O}_R(P) &
\end{tikzcd}$$

starting at $[A/R]_0 :\equiv A$ (or at $[A/R]_{-1} :\equiv \mathbf{0}$).

- Define $[I_R]_n :\equiv \mathsf{pr}_1 \circ [q_R]_n : [A/R]_n \to A \to \mathsf{U}$
- For each $n : \mathbb{N}$ and each $b : A$, there is an equivalence

$$\sum_{(x:[A/R]_n)} [I_R]_n(x, a) \simeq \Big(\sum_{(a:A)} R(a, b)\Big)^{*(n+1)}.$$

- The map

$$[q_R]_\infty : [A/R]_\infty \to \sum_{(P:\mathsf{im}(R))} \mathcal{O}_R(P)$$

is an equivalence.

We obtain a map $\mathcal{Q}_A$ from the type of principal equivalence relations on A to the type of surjective maps out of A, with small codomains.

Theorem

For each type A, the map $\mathcal{Q}_A$ is an equivalence.

The join of maps

$$
\begin{tikzcd}
A \times_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\mathsf{inr}"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\mathsf{inl}"'] \arrow[drr, bend right, "f"'] & A *_X B \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "f * g"'] & \\
 & & X
\end{tikzcd}
$$

Definition

For any $f : A \to X$, we define a sequence

$$\begin{tikzcd}
A_0 \arrow[r, "i_0"] \arrow[dr, "f^{*0}"'] & A_1 \arrow[r, "i_1"] \arrow[d, "f^{*1}"'] & A_2 \arrow[r, "i_2"] \arrow[dl, "f^{*2}"'] & \cdots \arrow[dll, "f^{*3}"] \\
& X & &
\end{tikzcd}$$

The function f^{*n} is called the *n-th join-power of f*.

Construction.

We take $A_0 :\equiv \mathbf{0}$, with the unique map into X. Then we define $A_{n+1} :\equiv A_n *_X A$, and $f^{*(n+1)} :\equiv f^{*n} * f$. $\square$

- In the real projective case:
  - $\mathbb{R}\mathsf{P}^{n+1} = \mathbb{R}\mathsf{P}^{n} *_{\mathbb{R}\mathsf{P}^\infty} \mathbf{1}$, and
  - $\mathrm{cov}_2^n = (\mathrm{cov}_2^0)^{*(n+1)}$
- In the complex projective case:
  - $\mathbb{C}\mathsf{P}^{n+1} = \mathbb{C}\mathsf{P}^{n} *_{\mathbb{C}\mathsf{P}^\infty} \mathbf{1}$, and
  - $\mathrm{cov}_{\mathbb{S}^1}^n = (\mathrm{cov}_{\mathbb{S}^1}^0)^{*(n+1)}$
- In the general projective case:
  - $\mathsf{P}(X)^{n+1} = \mathsf{P}(X)^{n} *_{\mathbf{B}X} \mathbf{1}$, and
  - $\mathrm{cov}_X^n = (\mathrm{cov}_X^0)^{*(n+1)}$

Theorem

The sequential colimit $f^{*\infty}$ is an embedding, and has the universal property of the image inclusion

$$A \to \mathsf{im}(f) \to X$$

Corollary

The sequential colimit of the type sequence

$$\mathbf{0} \longrightarrow A \xrightarrow{\mathsf{inr}} A * A \xrightarrow{\mathsf{inr}} A * (A * A) \xrightarrow{\mathsf{inr}} \cdots$$

is the propositional truncation $\|A\|_{-1}$.

Definition

A type X, which may itself be large, is said to be locally small if for all $x, y : X$, there is a type $x =' y : \mathrm{U}$ and an equivalence of type

$$(x = y) \simeq (x =' y).$$

Examples:

- all types in U,
- the universe U,
- mere propositions of any size,
- for any $A : \mathrm{U}$ and any locally small type X, the type $A \to X$.

The modified join of maps into locally small X

Take $A \times'_X B :\equiv \sum_{(a:A)} \sum_{(b:B)} f(a) =' g(b)$.

$$
\begin{tikzcd}
A \times'_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\mathrm{inr}"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\mathrm{inl}"'] \arrow[drr, bend right, "f"'] & A *'_X B \arrow[dr, dotted, "f *' g"'] & \\
 & & X
\end{tikzcd}
$$

Theorem

Assumptions:

- U *is a univalent universe closed under pushouts,*
- *let* X *be a locally small type,*
- *let* $f : A \to X$ *with* $A : \mathsf{U}$*,*

Then we can construct a type $\mathrm{im}'(f) : \mathsf{U}$*, a surjective map* $q'_f : A \to \mathrm{im}'(f)$*, and an embedding* $i'_f : \mathrm{im}'(f) \to X$ *such that the triangle*

$$\begin{tikzcd} A \arrow[r, "q'_f"] \arrow[dr, "f"'] & \mathrm{im}'(f) \arrow[d, "i'_f"] \\ & X \end{tikzcd}$$

commutes, with the universal property of the image inclusion of f*.*

level	*equivalence structure*	*quotient operation*
-2	trivial relation	propositional truncation
-1	Prop-valued equivalence relation	set-quotient
0	pre-1-groupoid structure	Rezk completion
$\vdots$	$\vdots$	$\vdots$
∞	‘pre-∞-groupoid structure’	∞-quotient

- For any modality $\bigcirc$, the modality of $\bigcirc$-separated types.
- n-truncation in any univalent universe closed under pushouts.
- $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, ...