General introduction to the theory of magnetohydrodynamics

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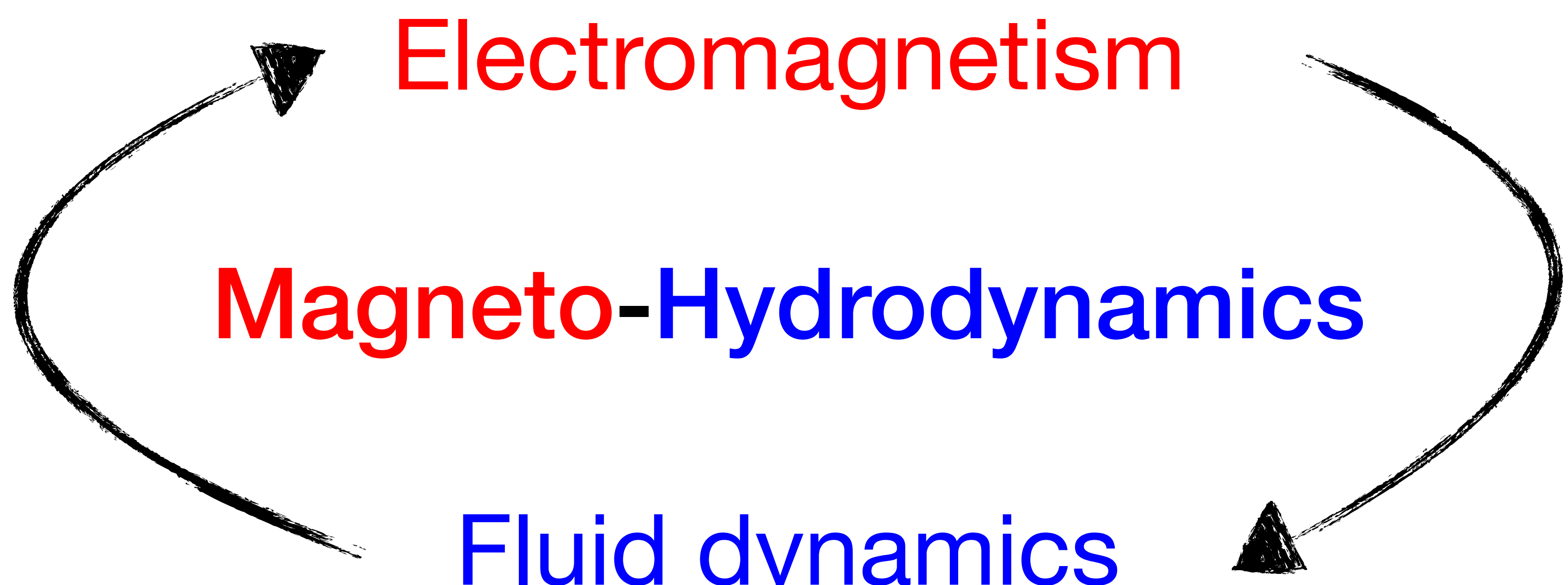

"A mechanical motion in the liquid will in general give rise to an e.m.f., which produces electric currents.

The interaction between the magnetic field and these currents causes mechanical forces which change the state of motion of the liquid."

"As the term electromagnetic-hydrodynamic is somewhat complicated, it may be convenient to call the phenomenon magnetohydrodynamic. (The term hydromagnetic is still shorter but not quite adequate.)"

1666 – 1687 Leibnitz & Newton
development and publication of infinitesimal calculus

1757 Euler
continuity & momentum equations

1816 Laplace
fluid energy equation

1821 Navier
momentum equation with viscosity

1820 Ørsted
electric current deflects compass

1821 Faraday
motor experiment

1826 Ampère
“Mathematical Theory of Electrodynamic Phenomena”

1831 Faraday
induction experiments

?

Anemotaxis in Drosophila

COLE[1] has observed that *Drosophila melanogaster* sometimes walks against an air current. Flügge[2] showed that this reaction only occurred when the air was scented, and must therefore be regarded as orientation by smell. However, reactions to air currents without smell do occur in Drosophila.

Comparing several species of the genus, a division can be made between those which show a distinct positive anemotaxis and those which do *not*. *Drosophila virilis* +, *D. virilis americana*, *D. subobscura* and *D. funebris* turn very sharply towards a tube from which air is flowing and start walking against it, so long as they are not blown away. *D. melanogaster* and *D. busckii*, on the other hand, show no reaction, whereas *D. pseudo-obscura* shows a slight reaction.

Removal of the wings or antennae or both does not abolish the reaction in the species showing it. Therefore Fraenkel and Gunn's[3] contention that anemotaxis depends on the perception of body deformation caused by the wind seems plausible for Drosophila.

The fact that the two light species observed do not react to wind whereas the four dark species do, may be an indication that the former are not exposed to strong air currents in their natural environment, while the latter are. As wind is a factor which increases evaporation, this is in agreement with some recently published deductions[4], to the effect that a dark cuticle provides better protection against desiccation than a light one.

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[1] Cole, W. H., *J. Anim. Behaviour*, 7, 71 (1917).
[2] Flügge, C., *Z. vergl. Physiol.*, 20, 463 (1934).
[3] Fraenkel and Gunn, "Orientation of Animals" (Oxford, 1940).
[4] Kalmus, H., NATURE, 148, 428 (1941).

Nomenclature of Biological Movement

HAVING read recently with very great interest Fränkel and Gunn's "Orientation of Animals", I set about making for my own clarification a classification of all the cases I know of what used to be called 'tropistic' movements.

It is clear that three major groupings are possible: a *taxis*, which is a bodily movement of an animal or motile plant in a direction determined by the direction of the stimulus; a *kinesis*, which is a change of rate of movement of an animal (or perhaps motile plant) in response to a change of intensity of a stimulus, but not in a direction determined by the direction of the stimulus—and often producing an aggregating effect superficially similar to that of a taxis; and an *orientation*, which is the placing of the body (usually if not always animal) in a direction determined by the direction of the stimulus. To these three classes many of the cases can be referred.

But the responses of sessile plant organs do not seem to be so conveniently classified. The thigmotropism of Clematis tendrils appears to warrant that name, for the response is a directional one. But the same cannot be said of the so-called 'thigmotropism' of Mimosa leaflets, Mimulus stigma or Berberis stamen, for here the response is not in a direction determined by that of the stimulus. The response in these cases appears to bear a closer resemblance to the photonasty of Oxalis leaflets and the thermonasty of Tulipa flowers. Is one then justified (disregarding—as often becomes necessary for purposes of coherence—mere etymological niceties) in putting these responses under the heading of 'thigmonasty' (or possibly in some cases—as in Stiles's "Plant Physiology"—'seismonasty')? By the same token may the response of Mimosa leaflets to a lighted match in the neighbourhood be called a 'thermonasty', and to ammonia vapour, a 'chemonasty'?

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An Inland Record of *Triglochin maritimum* L.

IN a previous note[1] I recorded the occurrence of the halophilic alga *Percursaria percursa* Rosenv. from a salt-spring situated at Aldersey[2], Cheshire. Continuing the ecological survey of this spring, I am now able to report the presence of the halophyte *Triglochin maritimum* L. (Naiadaceae). It occurs in three small colonies growing in the salt-spring and appears to be well established. I should add that during the past year the spring has maintained a salinity of 1,642 parts per 100,000 with only slight variation.

Triglochin maritimum, popularly known as seaside arrow-grass, is a plant frequently found in salt marshes. Its occurrence in a non-littoral region is exceptional and would seem to be unique in so far as the county of Cheshire is concerned. Search of the literature has revealed the fact that Triglochin has occasionally been recorded from the counties of Cambridgeshire, Staffordshire and Surrey.

In Cambridgeshire, the plant appears to have been first reported from Tydd Marsh by Skrimshire[3]. A. H. Evans[4], however, points out that it was never plentiful in that county and has not been found there during recent years. In Staffordshire, J. E. Bagnall[5] mentions records of Triglochin by the independent observers, Shaw, Stokes and Brown. In Surrey, C. E. Britton[6] records its discovery by W. A. Todd from the Thames near Putney.

I am indebted to Mr. A. A. Dallman, of Doncaster, for certain of the references quoted.

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Aug. 29.

[1] Burke, F., NATURE, 149, 331 (1942).
[2] Sherlock, Mem. Geol. Survey, Mineral Resources of Gt. Brit., Rock-salt and Brine, 18, 111 (1921).
[3] Relhan, R., "Flora Cantabrigiensis", second edition, 145 (1802).
[4] Evans, A. H., "Flora of Cambridgeshire", 165 (1939).
[5] Bagnall, J. E., "Flora of Staffordshire", 57 (1901).
[6] Britton, C. E., *J. Bot.*, 48, 186 (1910).

Existence of Electromagnetic-Hydrodynamic Waves

IF a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an E.M.F. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid.

Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = \frac{4\pi}{c}\, i$$

$$\text{rot } E = -\frac{1}{c}\,\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma\left(E + \frac{v}{c} \times B\right);$$

together with the hydrodynamic equation

$$\partial\,\frac{dv}{dt} = \frac{1}{c}\,(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, ∂ the mass density of the liquid, i the electric current, v the velocity of the liquid, and p the pressure.

Consider the simple case when $\sigma = \infty$, $\mu = 1$ and the imposed constant magnetic field H_0 is homogeneous and parallel to the z-axis. In order to study a plane wave we assume that all variables depend upon the time t and z only. If the velocity v is parallel to the x-axis, the current i is parallel to the y-axis and produces a variable magnetic field H' in the x-direction. By elementary calculation we obtain

$$\frac{d^2H'}{dz^2} = \frac{4\pi\partial}{H_0^2}\,\frac{d^2H'}{dt^2},$$

which means a wave in the direction of the z-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi\partial}}.$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 15$ gauss and $\partial = 0{\cdot}005$ gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and ∂ refer to a distance of about 10^{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

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Energy of Dissociation of Carbon Monoxide

THE energies of dissociation of a number of diatomic molecules have been determined from spectroscopic data, apparently with high accuracy, by the observation of predissociation limits. During the last few years the following values have been proposed for CO: $D(\text{CO}) = 6{\cdot}92$ [1], $8{\cdot}41$ [2], $9{\cdot}14$ [3], $10{\cdot}45$ [4] e.v.; while values of $9{\cdot}85$ and $11{\cdot}11$ also appear possible[5]. Controlled electron experiments suggest $9{\cdot}6$ [5].

The value obtained by extrapolation of the vibrational levels of the ground state is about 11, and support for this value has been given by Kynch and Penney[6]. Herzberg[7] has recently summarized evidence favouring 9·14.

At first sight, the strongest argument for 9·14 is the observation by Faltings, Groth and Harteck[8] that CO is decomposed by the xenon line at 1295 A., but not by that at 1470 A., from which they conclude that $8{\cdot}44 < D(\text{CO}) < 9{\cdot}57$. This conclusion is not based on an examination of the initial act of absorption. The only known absorption in the 1295 A. region is that corresponding to the fourth positive bands. The origins of the (9,0) and (10,0) bands lie at 76,839 cm.$^{-1}$ and 78,010 cm.$^{-1}$. The xenon line 1295 A. = 77,172 cm.$^{-1}$ falls between these bands and, if absorbed from the lowest vibrational level of CO, would correspond approximately to the line $P(35)$ of (10,0). This gives as the upper limit of $D(\text{CO})$ (when the rotational energy is taken into account) a value of 79,722 cm.$^{-1}$ = 9·88 e.v. (not 9·57 e.v. as stated by Herzberg[7]). Actually, it is doubtful whether such a high rotational line as $P(35)$ would be observed at room temperature, and absorption, if it is due to CO, would probably occur from a higher vibrational level, corresponding perhaps to the (13,2) band, in which case the dissociation limit may be placed as high as 10·1.

Taking the first act of absorption as

$$\text{CO}(X^1\Sigma) + h\nu = \text{CO}(A^1\Pi),$$

and assuming a life not less than 10^{-8} sec. for $A^1\Pi$, then at atmospheric pressure each molecule experiences at least 100 collisions before radiating. It seems to us that this gives a reasonable chance for a reaction such as

$$\text{CO}(A^1\Pi) + \text{CO}(X^1\Sigma) = \text{CO}_2 + \text{C}$$

to proceed with quantum efficiency approaching unity. The state of the carbon atom might be either ^1D or ^3P: the former if spin is to be conserved, the latter if not. In either case the reaction is strongly exothermic. The failure of the xenon line 1470 to induce photodissociation may be due to the reaction requiring an activation energy.

Estimates of $D(\text{CO})$ less than 10 take no account of the non-crossing rule of Hund, and Neumann and Wigner[9]. This rule states that potential energy curves of molecular states of identical species cannot cross. Whether the rule is rigorous when the nuclear and electronic motions are not separated needs further examination, but at least we see no reason for anticipating a failure of the rule in the lowest energy curve of CO. If this curve has only one turning point then the non-crossing rule requires unequivocally that $D(\text{CO}) > 10{\cdot}3$, and would agree well with the predissociation limit at 11·11 e.v.

The dissociation energy of CO$^+$ is 2·6 e.v. less than that of CO ($D(\text{CO}^+) = D(\text{CO}) + I(\text{C}) - I(\text{CO})$). Three electronic states of CO$^+$ are known, namely, $X^2\Sigma^+$, $A^2\Pi$ and $B^2\Sigma^+$, extrapolating to dissociation limits of about 9·8 (a very long extrapolation), 9·2 and 9·4 e.v. respectively. Since the two $^2\Sigma^+$ states must give different products of dissociation, it would appear, on the evidence of the $B^2\Sigma^+$ state, that $D(\text{CO}^+)$ is 7·4, and $D(\text{CO})$ is about 10, and on the evidence of the $A^2\Pi$ state that $D(\text{CO}^+)$ is 9·2 and $D(\text{CO})$ is 11·8. All that may fairly be deduced from present evidence on CO$^+$ is that $D(\text{CO})$ is unlikely to be much less than 10.

We have also re-examined nitrogen. The accepted value $D(N_2) = 7{\cdot}38$ is based on the identification of the upper state of the Vegard–Kaplan bands with the

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Ingredients

Electromagnetism

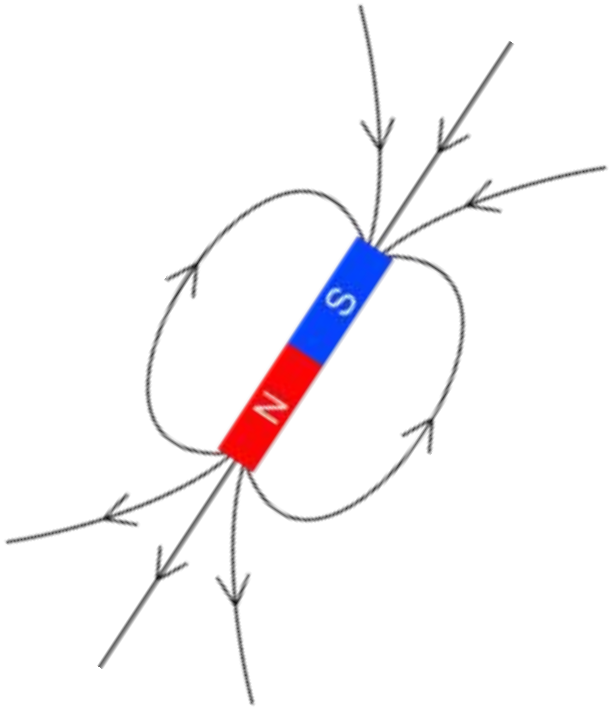
Gauss' law

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}$$

electric fields spread out from charged objects

Solenoidal constraint

$$\nabla \cdot \mathbf{B} = 0$$

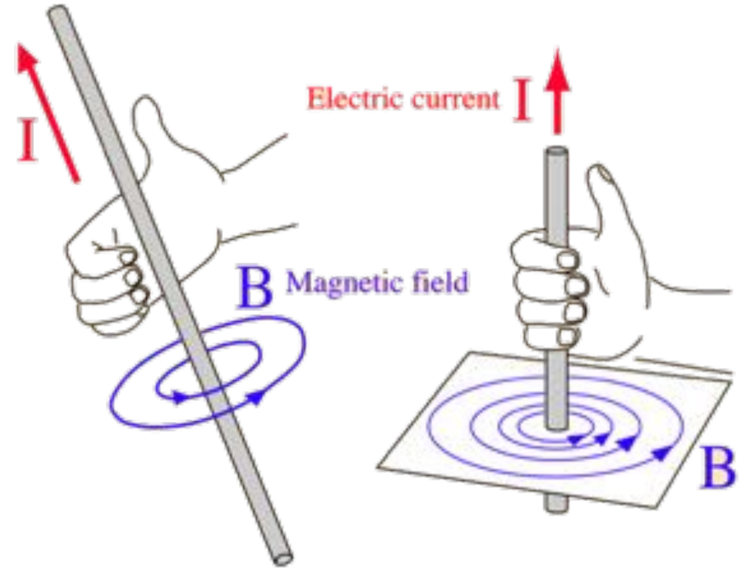


magnetic fields don't spread out from anything
no magnetic monopoles in classical E.M.

Ampère's law

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$



magnetic fields wrap around electric currents
and changing electric fields (displacement currents)

Faraday's law

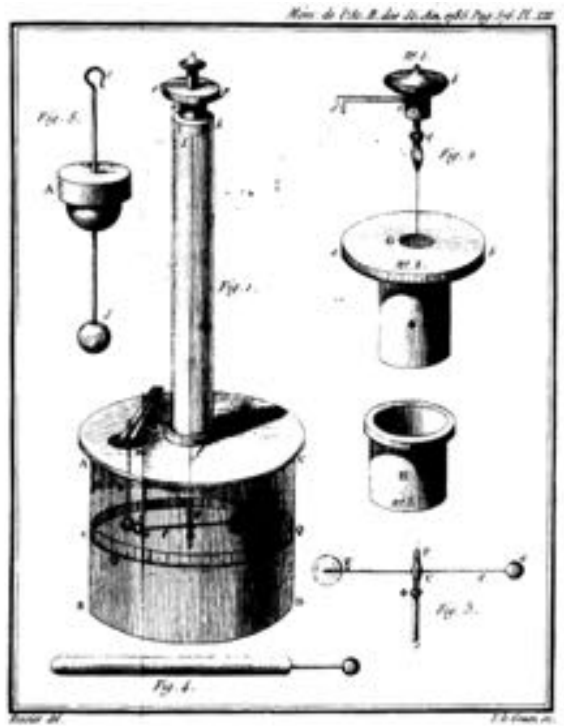
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

electric fields wrap around changing magnetic fields

Symmetries

	$\mathbf{E}$	$\mathbf{B}$
Div	$\nabla \cdot \mathbf{E} = \dfrac{\rho_c}{\epsilon_0}$	$\nabla \cdot \mathbf{B} = 0$
Curl	$\nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \mathbf{B} - \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$

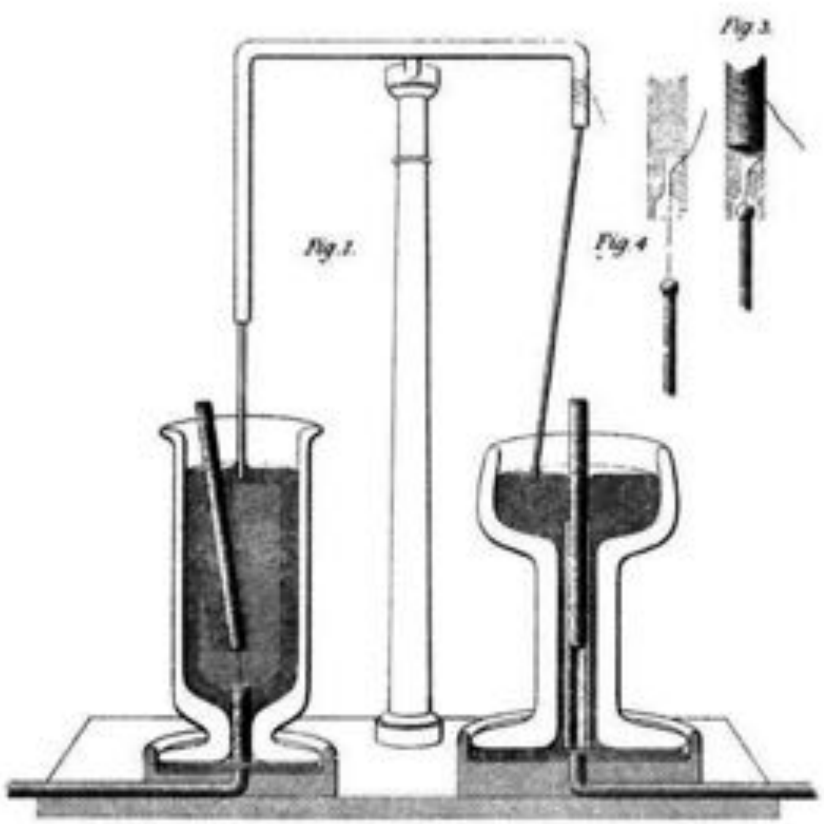
Electromagnetic forces (motors)



Electrostatic force per unit volume

$$\mathbf{F}_C = \rho_c \mathbf{E}$$

(from Coulomb's experiments, 1784)



Lorentz force per unit volume from a current

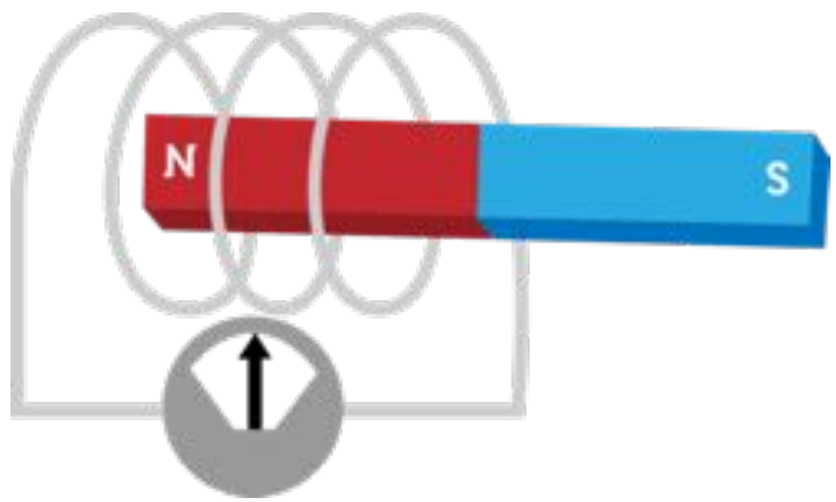
$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$$

(from Ampère's and Faraday's experiments, 1821-23)

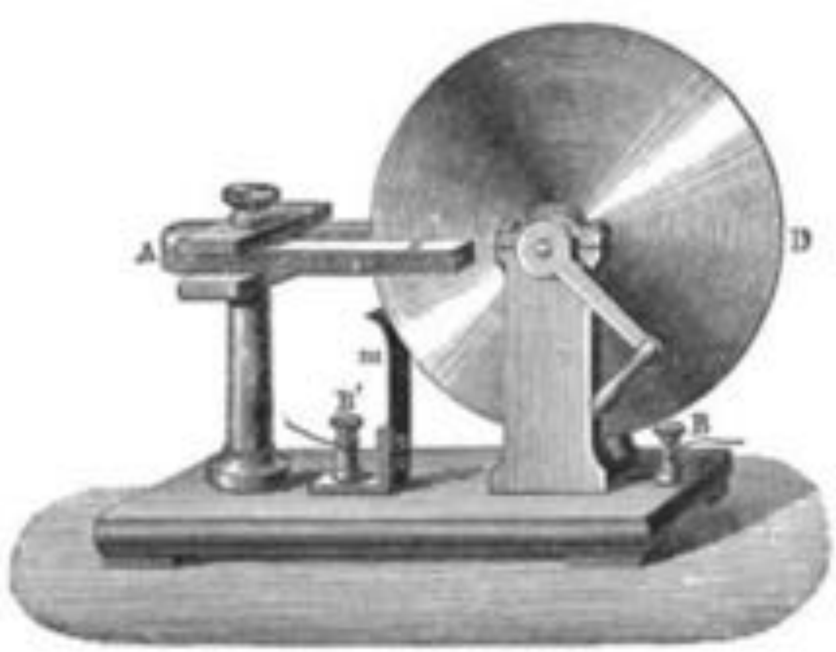
Ohm's law for conductors (generators)

For many materials on a lab bench, observe $\mathbf{j} = \sigma \mathbf{E}$

Induction:



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$



The more general relation that fixes the paradox is

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Electromagnetic energy and Poynting's theorem (1884)

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}}_{\text{energy of electric field plus energy of magnetic field}}\right) = \epsilon_0 \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}$$

by Faraday's and Ampère's

$$= \epsilon_0 \mathbf{E}\cdot\frac{1}{\mu_0\epsilon_0}(\nabla\times\mathbf{B} - \mu_0\mathbf{j}) + \frac{1}{\mu_0}\mathbf{B}\cdot(-\nabla\times\mathbf{E})$$

$$= -\mathbf{E}\cdot\mathbf{j} + \frac{1}{\mu_0}(\mathbf{E}\cdot\nabla\times\mathbf{B} - \mathbf{B}\cdot\nabla\times\mathbf{E})$$

by vector id

$$= -\mathbf{E}\cdot\mathbf{j} - \nabla\cdot\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)$$

$$\boxed{\frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j} \qquad U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}}$$

Electromagnetic energy and Poynting's theorem (1884)

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j} \qquad U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}}$$

How is energy converted with Ohm's law?

$$\mathbf{j} = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \quad \Leftrightarrow \quad \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{\mathbf{j}}{\sigma}$$

work done
on conductor

$$-\mathbf{E} \cdot \mathbf{j} = \mathbf{v} \times \mathbf{B} \cdot \mathbf{j} - \frac{j^2}{\sigma} = \mathbf{v} \cdot \mathbf{B} \times \mathbf{j} - \frac{j^2}{\sigma} = -\mathbf{v} \cdot \mathbf{j} \times \mathbf{B} - \frac{j^2}{\sigma}$$

resistive
heating

Summary of key E.M. results

Forces

$$\mathbf{F}_C = \rho_c \mathbf{E}$$

$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$$

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

Poynting's theorem

$$U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \qquad \frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}$$

Ohm's law for a conductor

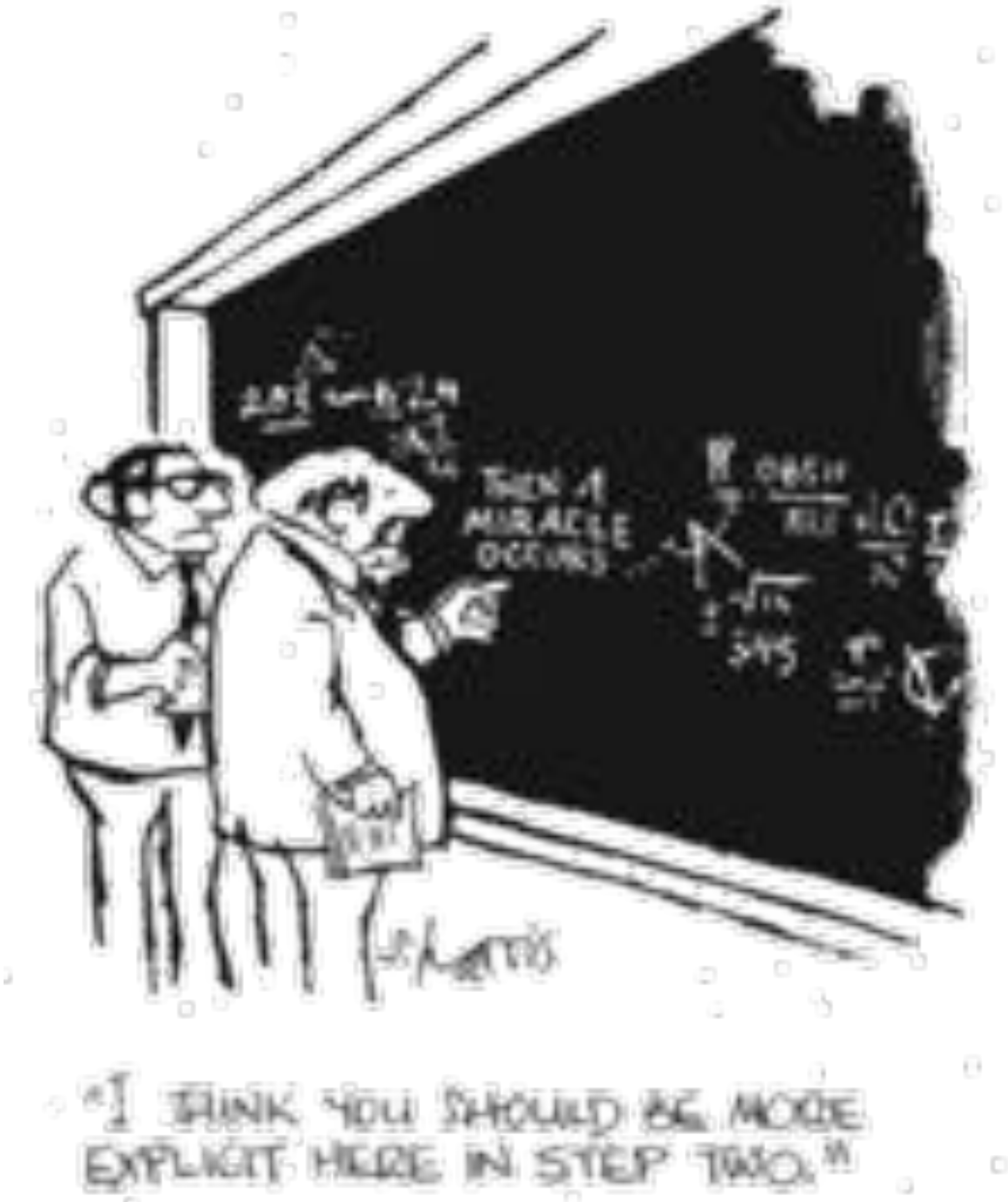
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

Heating

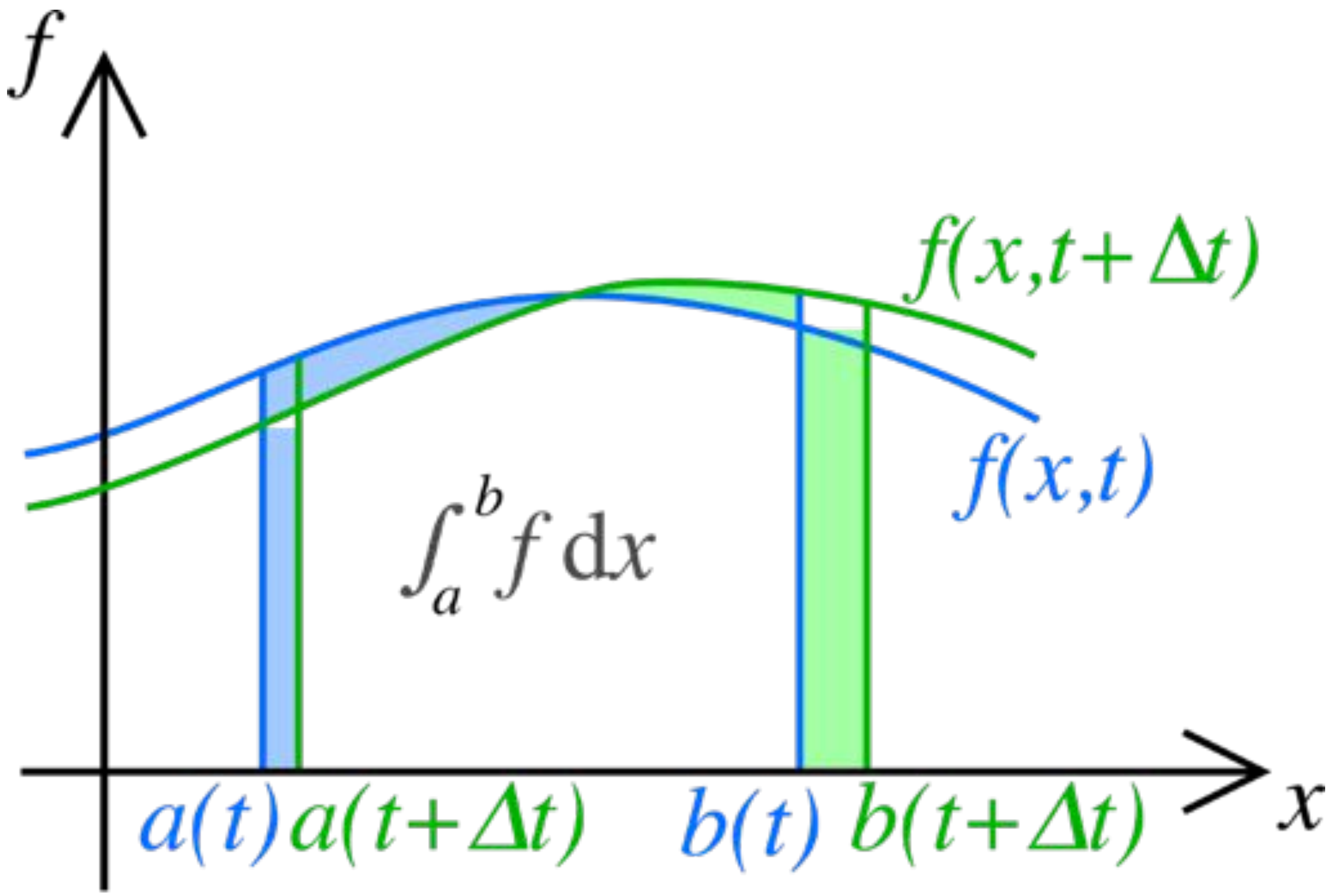
$$Q_{\mathrm{ohmic}} = \frac{j^2}{\sigma}$$

Fluid equations

Some useful calculus

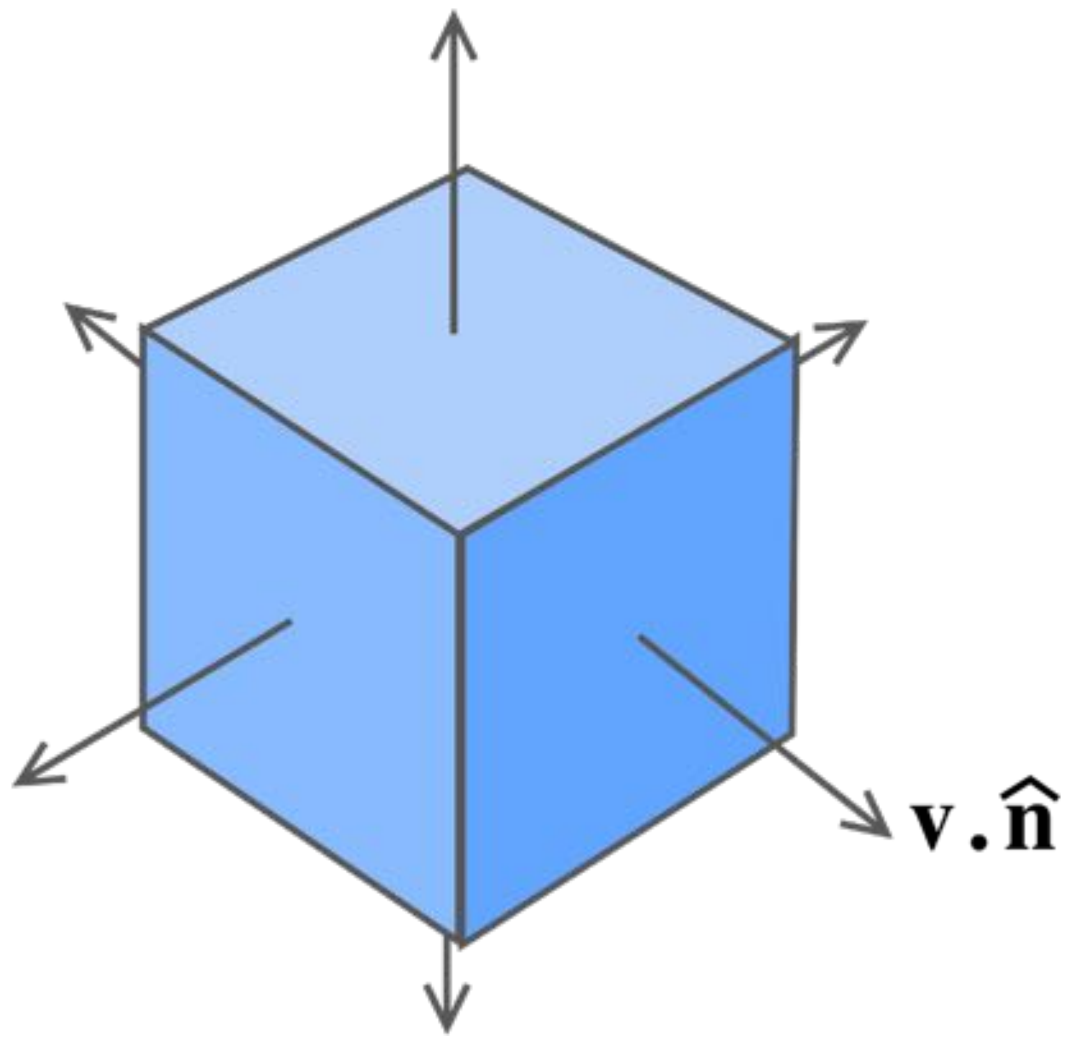


Leibnitz rule for 1-dimensional space



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x + f(b)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a)\frac{\mathrm{d}a}{\mathrm{d}t}$$

Leibnitz rule for 3-dimensional space



$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f(\mathbf{x},t)\,\mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t}\,\mathrm{d}V + \iint_{\partial V(t)} f\,\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

$$= \iiint_{V(t)} \left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{v})\right)\mathrm{d}V$$

By divergence theorem

Gauss theorems

$$\iint_{\partial V} \hat{\mathbf{n}} \star \mathcal{A} \, \mathrm{d}S = \iiint_V \nabla \star \mathcal{A} \, \mathrm{d}V$$

When $\mathcal{A}$ is a vector field and product is the dot product, get the divergence theorem of vector calculus.

When $\mathcal{A}$ is a scalar field and product is scalar multiplication, get the gauss theorem for a gradient.

Continuity equation (mass conservation)

Mass conservation: Classically, matter is neither created nor destroyed.

Define the mass density as

$$\rho(\mathbf{x},t) = \lim_{\delta V \to 0} \frac{\delta M}{\delta V} \qquad M(V,t) = \iiint_V \rho(\mathbf{x},t)\,\mathrm{d}V$$

Using the 3-d Leibnitz theorem with boundary moving with the fluid

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \iiint_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \mathrm{d}V = 0$$

by conservation of mass

If true for any volume of fluid, then everywhere

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

Convective derivative

The total time derivative when moving with the fluid is called the *convective derivative*. Using the chain rule it is given by

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Capital $\mathbf{D}$ used to signify special choice for velocity

Contribution from local changes in time

Contribution from spatial derivatives

A very useful result

Starting with the 3-d Leibnitz theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho a \,\mathrm{d}V = \iiint_V \left(\frac{\partial}{\partial t}(\rho a) + \nabla\cdot(\rho a \mathbf{u})\right)\mathrm{d}V$$

$$= \iiint_V \left(a\cancel{\frac{\partial \rho}{\partial t}} + \rho\frac{\partial a}{\partial t} + \cancel{a\nabla\cdot(\rho\mathbf{u})} + \rho\mathbf{u}\cdot\nabla a\right)\mathrm{d}V$$

$$= \iiint_V \rho\frac{\mathrm{D}a}{\mathrm{D}t}\,\mathrm{d}V$$

$$f = \rho a$$
$$\mathbf{v} = \mathbf{u}$$

continuity equation says the cancelled terms sum to zero

Since this applies to each component of $\rho\mathbf{A}$, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{A}\,\mathrm{d}V = \iiint_V \rho\frac{\mathrm{D}\mathbf{A}}{\mathrm{D}t}\,\mathrm{d}V$$

Momentum equation (Newton's 2nd law)

Newton's 2nd law: The rate of change of an object's momentum is equal to the sum of the forces acting on the object ($\mathbf{F} = m\mathbf{a}$).

The momentum per unit mass is the velocity, so

$$\text{sum of forces on } V = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\mathbf{u}\,\mathrm{d}V = \iiint_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V$$

Consider two types of forces:

- *Body forces* act within the volume, e.g. gravity.
- *Contact forces* act on the surface that bounds the volume.

Momentum equation (Newton's 2nd law)

Body forces

Introduce $\mathbf{F}_b$ as the body force per unit volume, e.g. gravity contributes

$$\mathbf{F}_g = \lim_{\delta V \to 0} \left\{ \frac{\mathbf{g}\delta M}{\delta V} \right\} = \rho \mathbf{g}$$

The total body force acting on the volume will therefore be

$$\iiint_V \mathbf{F}_b \, \mathrm{d}V$$

Contact forces

Total is determined by integrating contact force per unit area over the bounding surface.

If the only contact force is pressure acting normal to the surface, $-p\hat{\mathbf{n}}$, the total contact force on V is

$$\iint_{\partial V} -p\hat{\mathbf{n}} \, \mathrm{d}S = \iiint_V -\nabla p \, \mathrm{d}V$$

Momentum equation (Newton's 2nd law)

Matching the two sides of Newton's 2nd law therefore gives

$$\iiint_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V = \iiint_V \left(\mathbf{F}_b - \nabla p\right)\,\mathrm{d}V$$

If true for any volume of fluid, then everywhere

$$\boxed{\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p}$$

Note: Often need a more general treatment of contact forces.
An important case is the Navier-Stokes equation with viscosity

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p + \mu \left(\nabla^2 \mathbf{u} + \frac{1}{3} \nabla \left(\nabla \cdot \mathbf{u} \right) \right)$$

Energy equation

Our model conserves mass and momentum. Does it conserve energy?

<u>Rates at which forces do work</u>

Body forces $\iiint_V \boxed{\mathbf{u}\cdot\mathbf{F}_b}\,\mathrm{d}V$

Contact forces

$$\iint_{\partial V} \mathbf{u}\cdot(-p\hat{\mathbf{n}})\,\mathrm{d}S$$

$$= \iint_{\partial V} (-p\mathbf{u})\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

$$= \iiint_V \nabla\cdot(-p\mathbf{u})\,\mathrm{d}V$$

$$= \iiint_V \boxed{\mathbf{u}\cdot(-\nabla p)}\,\boxed{-\,p\nabla\cdot\mathbf{u}}\,\mathrm{d}V$$

<u>Change of kinetic energy</u>

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\frac{u^2}{2}\,\mathrm{d}V$$

$$= \iiint_V \rho\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{u^2}{2}\right)\mathrm{d}V$$

$$= \iiint_V \mathbf{u}\cdot\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V$$

$$= \iiint_V \mathbf{u}\cdot(\mathbf{F}_b - \nabla p)\,\mathrm{d}V$$

$$= \iiint_V \boxed{\mathbf{u}\cdot\mathbf{F}_b} + \boxed{\mathbf{u}\cdot(-\nabla p)}\,\mathrm{d}V$$

Work of compressing the volume is unaccounted for!

Energy equation (1st law of thermodynamics)

Fix: include the fluid's internal energy

Introduce ϵ as the specific internal energy density, i.e. per unit mass.

$$\text{Internal energy of a volume of fluid} = \iiint_V \rho\epsilon \,\mathrm{d}V$$

The first law says internal energy changes through work and heating.

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\epsilon\,\mathrm{d}V = \iiint_V \rho\frac{\mathrm{D}\epsilon}{\mathrm{D}t}\,\mathrm{d}V = -\iiint_V p\nabla\cdot\mathbf{u}\,\mathrm{d}V + \iiint_V Q\,\mathrm{d}V$$

heating rate per unit volume

If true for any volume of fluid, then everywhere

$$\boxed{\rho\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -p\nabla\cdot\mathbf{u} + Q}$$

Summary of fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\rho \frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -p \nabla \cdot \mathbf{u} + Q$$

How can we update the pressure?

Closure (LTE equation of state)

For an ideal fluid in local thermal equilibrium, i.e. assuming collisions have made the particle distribution a drifting Maxwellian, find:

$$\epsilon = \frac{s}{2}\left(\frac{k_B}{\bar{m}}\right)T \qquad p = \rho\left(\frac{k_B}{\bar{m}}\right)T$$

k_B is the Boltzmann constant
$\bar{m}$ is the average particle mass
s is the number of degrees of freedom

Strategy 1: Numerically, can calculate $p = \frac{2}{s}\rho\epsilon$ as needed.

Strategy 2:

$$\rho\frac{\mathrm{D}\epsilon}{\mathrm{D}t} = -p\nabla\cdot\mathbf{u} + Q \quad \Rightarrow$$

Eliminate

$$\epsilon = \frac{s}{2}\frac{p}{\rho}$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p\nabla\cdot\mathbf{u} + (\gamma-1)Q \qquad \gamma = \frac{2+s}{2}$$

or $$\rho^\gamma\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{p}{\rho^\gamma}\right) = (\gamma-1)Q$$

Summary of fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{F}_b - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1) Q$$

Where closure was by assuming LTE.

Single-fluid resistive MHD

Macroscopic method

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\epsilon = (\gamma - 1)\frac{p}{\rho} \qquad \gamma = \frac{s+2}{2}$$

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\rho}(\mathbf{F}_b - \nabla p)$$

$$T = \frac{p}{\mathcal{R}\rho} \qquad \mathcal{R} = \frac{k_B}{\bar{m}}$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1)Q$$

closure by assuming
ideal gas relations: LTE

E.M. equations

$$\mathbf{F}_C = \rho_c \mathbf{E} \qquad \nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\mathbf{F}_L = \mathbf{j} \times \mathbf{B} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma} \qquad Q_{\mathrm{ohmic}} = \frac{j^2}{\sigma}$$

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1)\frac{j^2}{\sigma}$$

1. After applying body forces and Ohmic heating rate

E.M. equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

2. Will set **v** = **u.**

3. Curl of Faraday's law implies div(**B**) remains zero, so don't need div(**B**) =0 as a separate condition.

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1)\frac{j^2}{\sigma}$$

4. Neglect displacement current and Coulomb force based on "quasi-neutrality". Dimensional analysis indicates this applies when phase and fluid speeds are much less than speed of light. Sometimes called the MHD approximation.

E.M. equations

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

At that point can drop Gauss' law.

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1)\frac{j^2}{\sigma}$$

5. Eliminate **j** using Ampère's law, and **E** using Ohm's law, which can then be dropped.

E.M. equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

Fluid equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \frac{(\gamma - 1)}{\mu_0^2 \sigma} |\nabla \times \mathbf{B}|^2$$

E.M. equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} \right)$$

MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \frac{(\gamma - 1)}{\mu_0^2 \sigma} |\nabla \times \mathbf{B}|^2$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} \right)$$

Assumptions used

- Fluid
- Isotropic pressure
- LTE
- Ohm's law
- Quasi-neutrality (velocities much less than c)

Magnetic forces

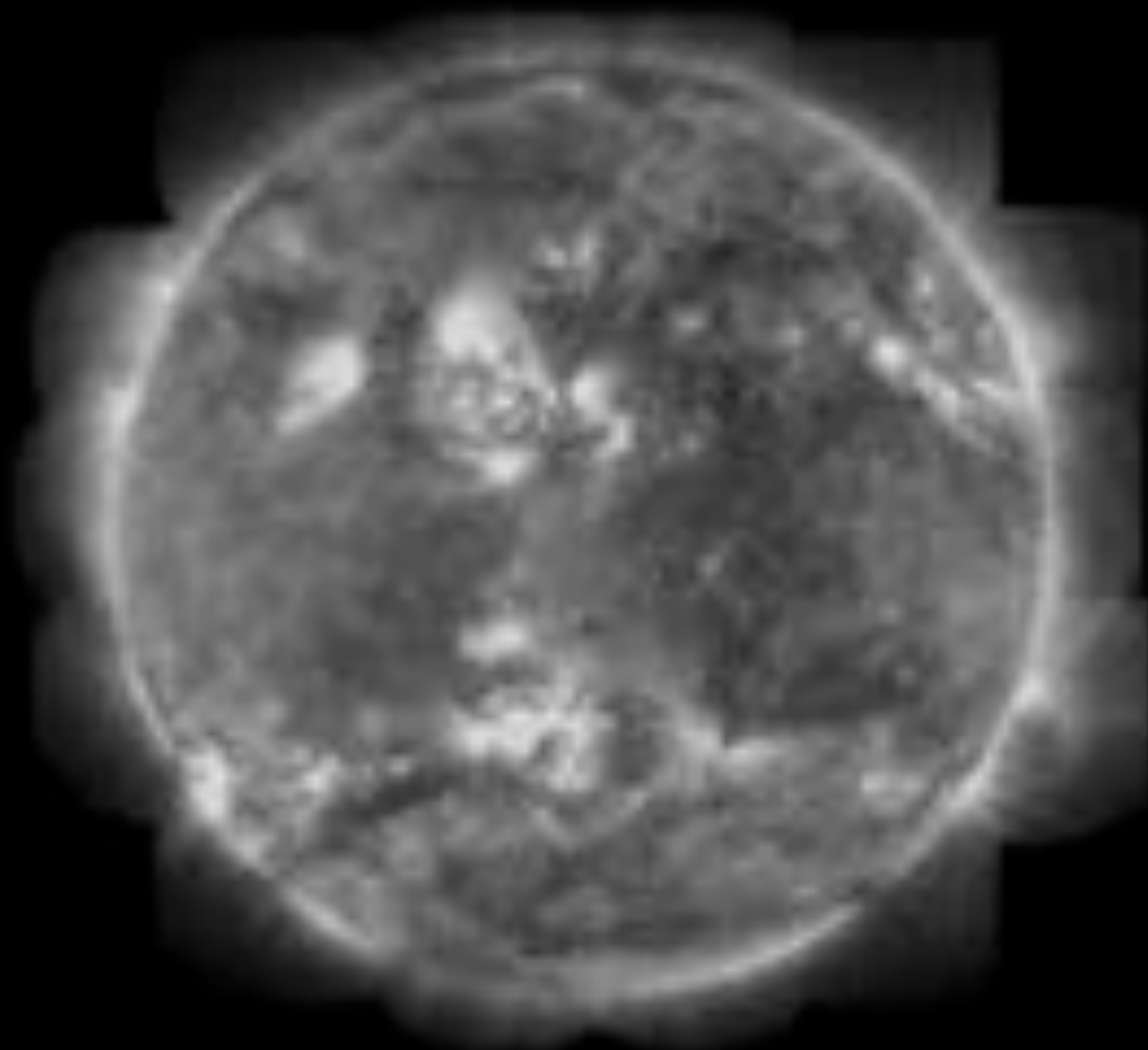
$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

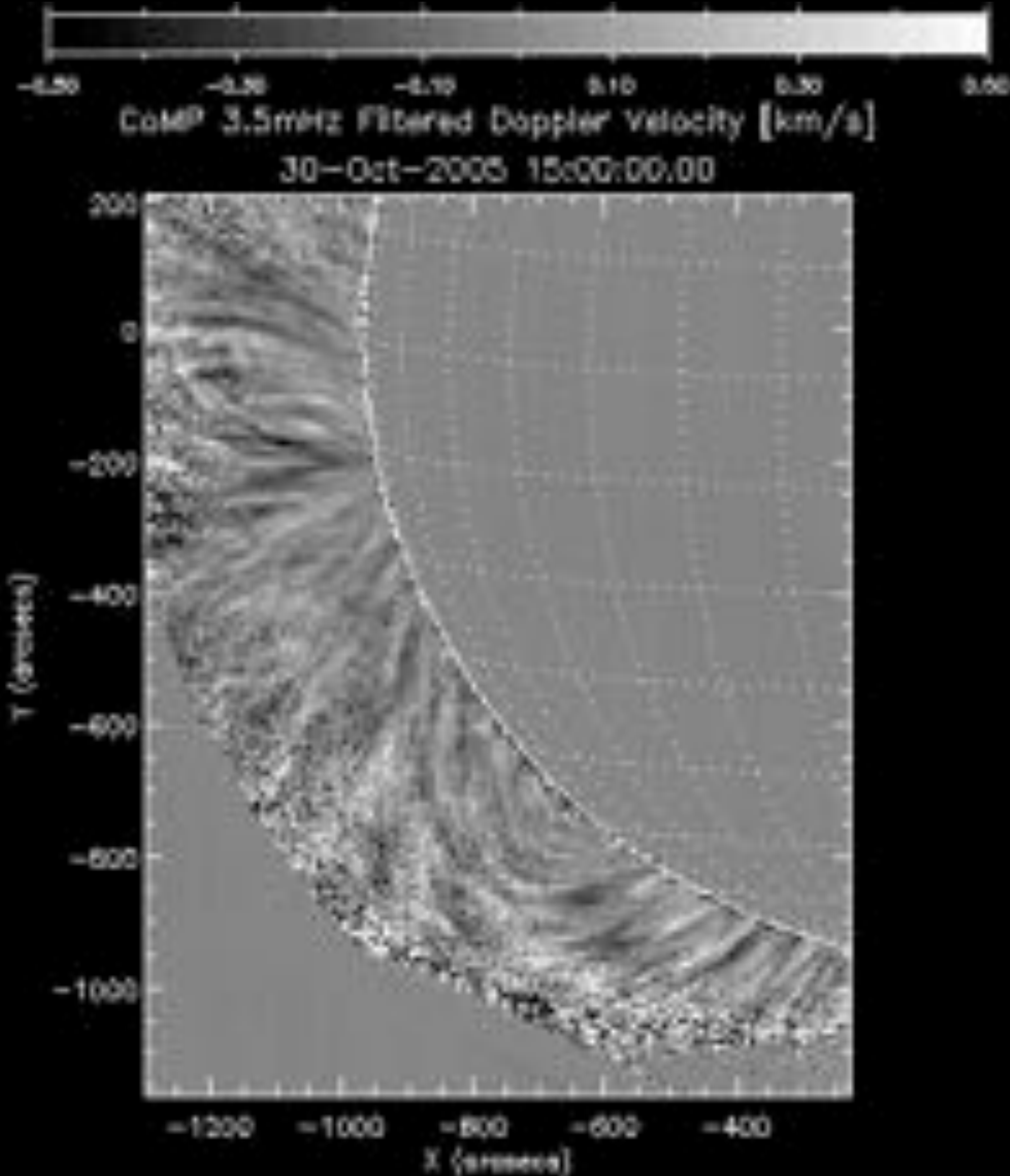
magnetic tension term

magnetic pressure term

The perpendicular part of the "tension" term is $\frac{B^2}{\mu_0} (\mathbf{e}_B \cdot \nabla) \mathbf{e}_B$



Standing coronal loop oscillations excited by flare (magnetic tension):
see http://adsabs.harvard.edu/abs/1999Sci...285..862N
http://adsabs.harvard.edu/abs/1999ApJ...520..880A



Detection of ubiquitous propagating waves in the corona (mag. tension: see http://adsabs.harvard.edu/abs/2007Sci...317.1192T

Magnetic forces

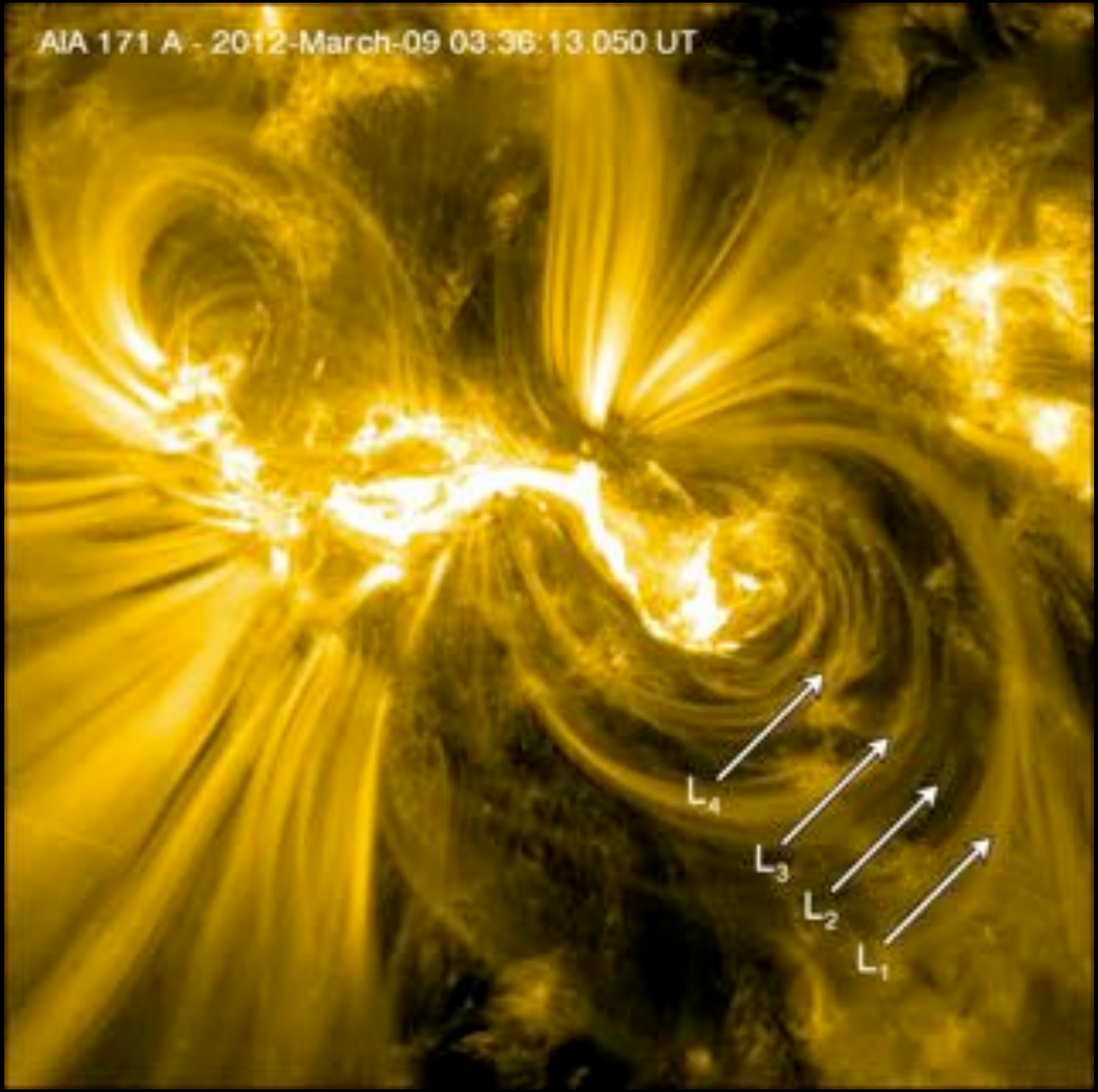
$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\,\mathbf{B} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

magnetic tension term

magnetic pressure term

Thermal pressure is proportional to internal energy per unit volume.
Magnetic pressure is is proportional to magnetic energy per unit volume.

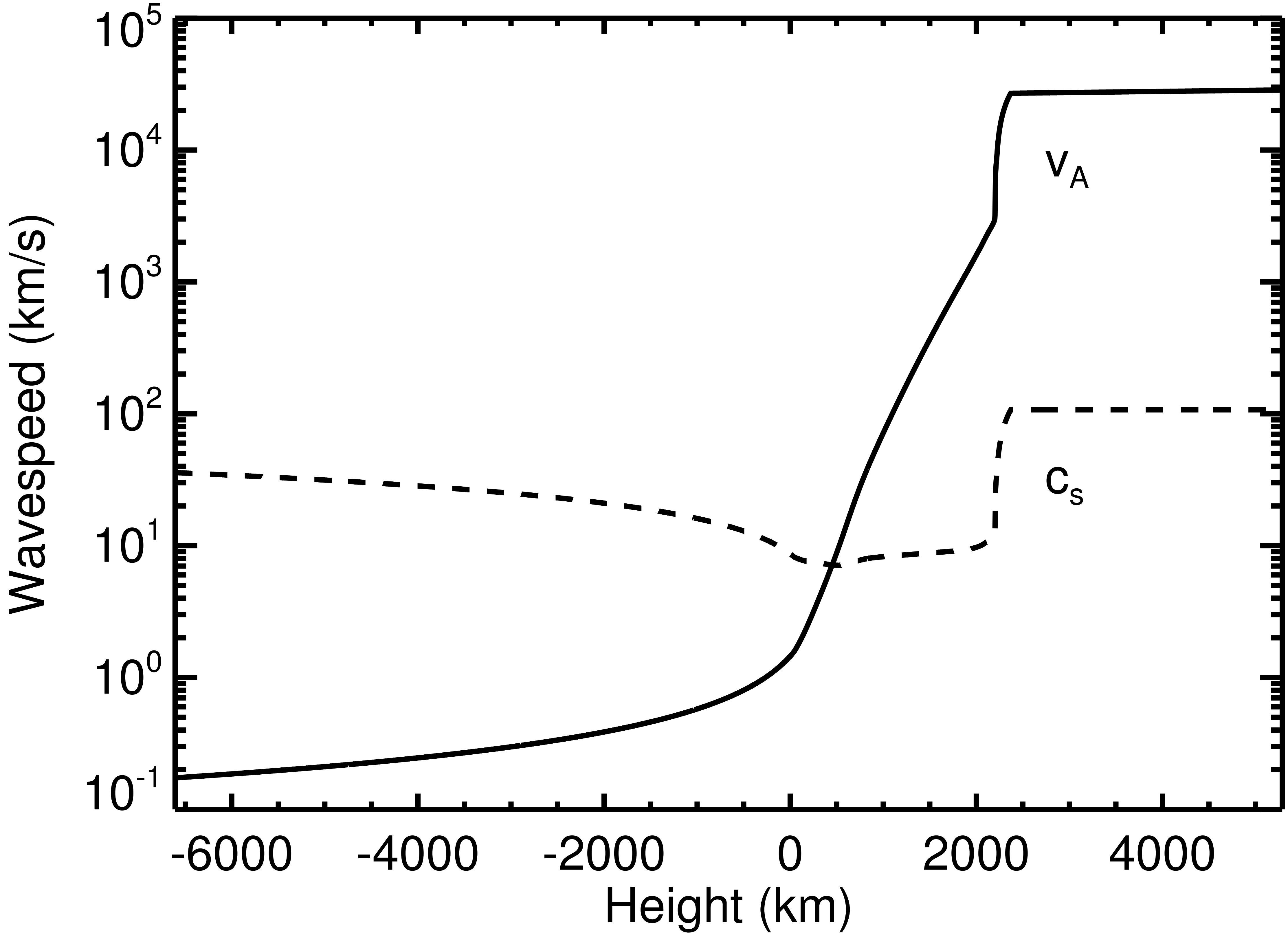


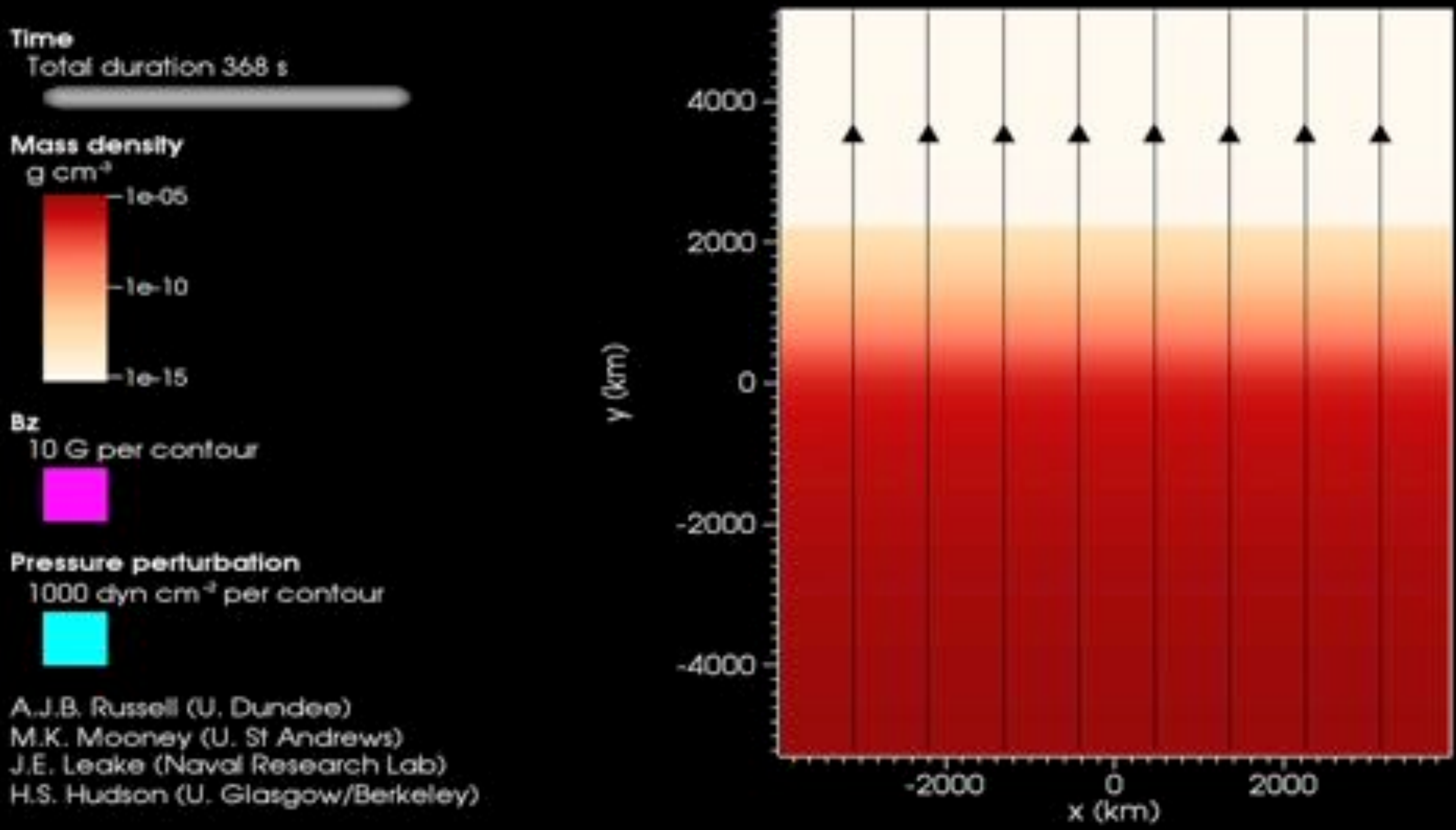
Coronal implosion (mag. pressure) with waves (mag. tension):
see http://adsabs.harvard.edu/abs/2015A%26A...581A...8R
http://adsabs.harvard.edu/abs/2017A%26A...607A...8P
http://adsabs.harvard.edu/abs/2013ApJ...777..152S

Thermal pressure forces

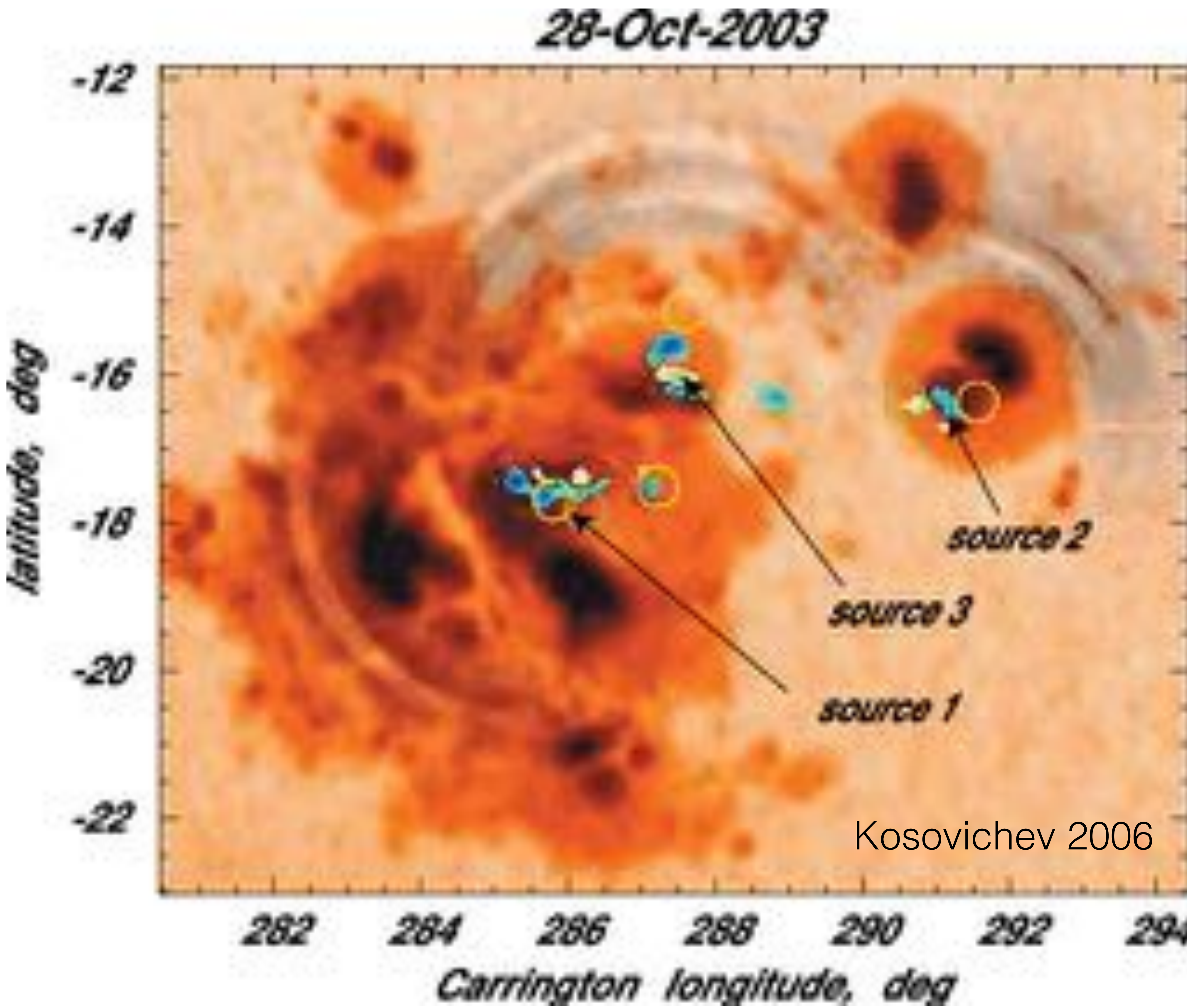
$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

Alfvén and sound speeds in the Sun





Sunquake generation by wave coupling at beta = 1:
see http://adsabs.harvard.edu/abs/2016ApJ...831...42R


28-Oct-2003
-12
-14
-16
-18
-20
-22
latitude, deg
282
284
286
288
290
292
294
Carrington longitude, deg
source 1
source 2
source 3
Kosovichev 2006

Induction equation

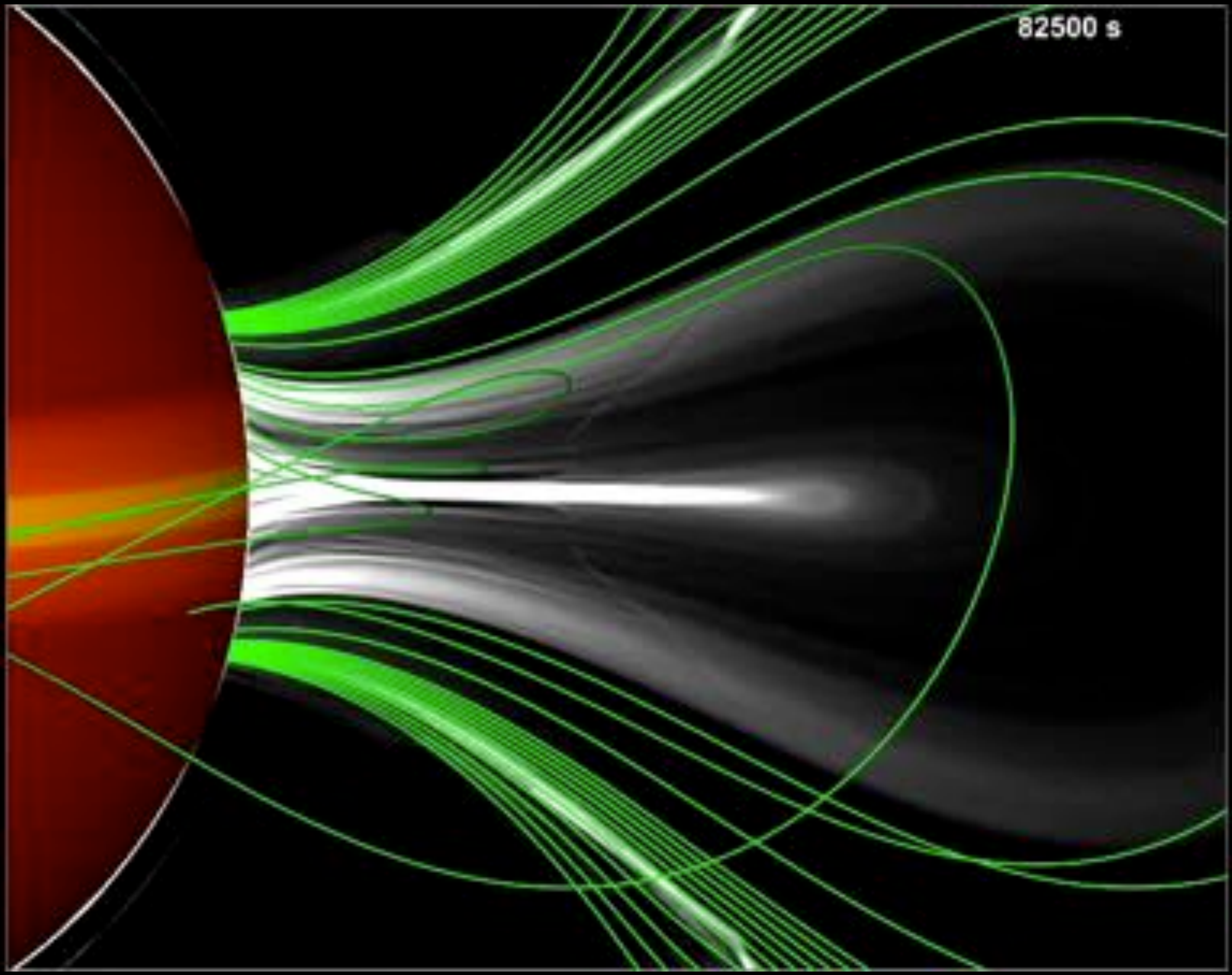
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad \eta = \frac{1}{\mu_0 \sigma}$$

High diffusion limit: $$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} - \nabla \eta \times \nabla \times \mathbf{B}$$

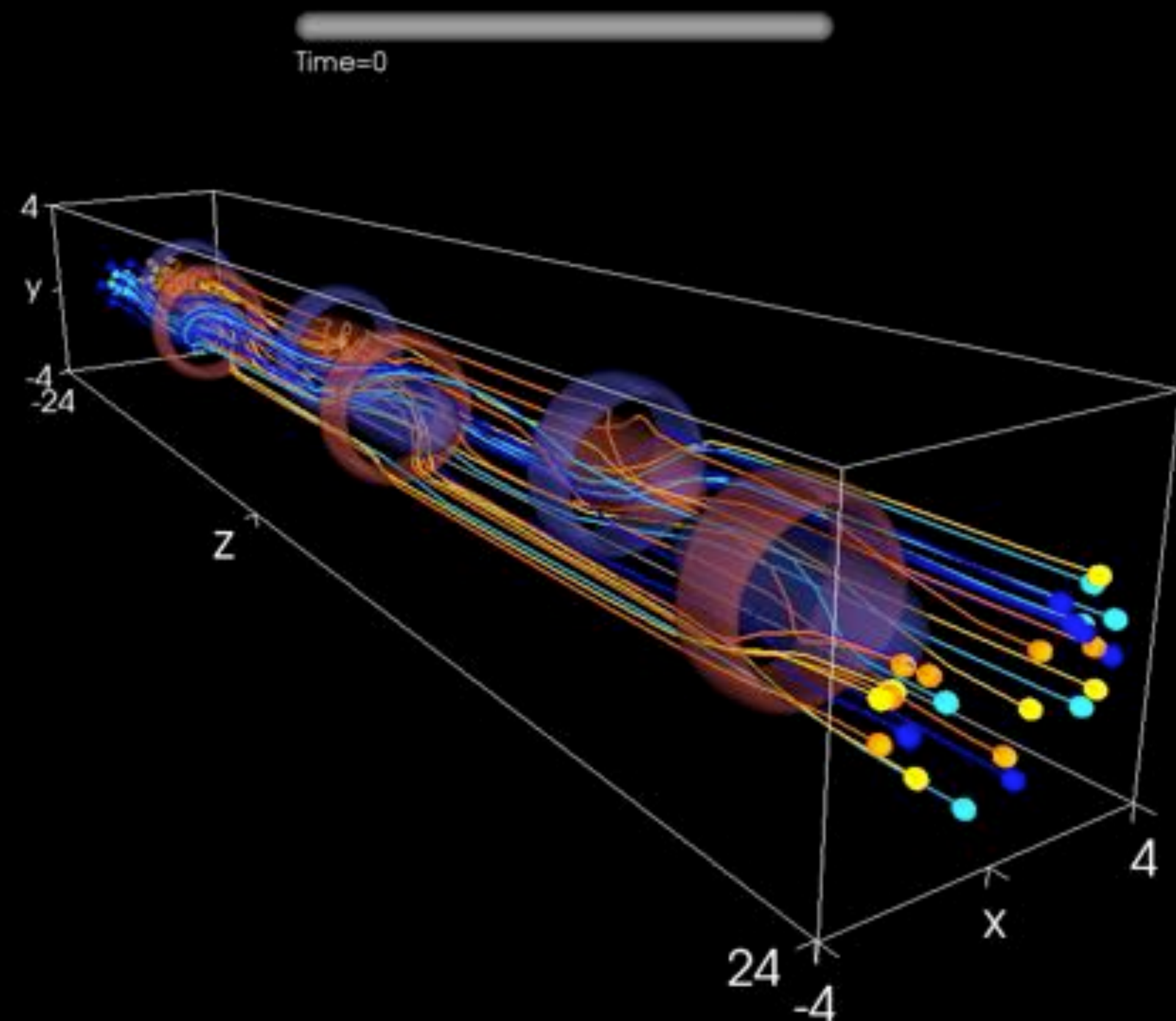
Ideal MHD limit: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} \,\mathrm{d}S &= \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \,\mathrm{d}S - \oint_{\partial S} \mathbf{u} \times \mathbf{B} \cdot \mathrm{d}\mathbf{l} \\ &= \iint_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \hat{\mathbf{n}} \,\mathrm{d}S - \iint_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot \hat{\mathbf{n}} \,\mathrm{d}S = 0.\end{aligned}$$

Magnetic field “frozen” to fluid. Magnetic topology conserved.



MHD simulation of fast triggered magnetic reconnection (high Rm): see http://adsabs.harvard.edu/abs/2012ApJ...760...81K



MHD simulation of a turbulently reconnecting magnetic braid
see http://adsabs.harvard.edu/abs/2016PPCF...58e4008P
http://www.maths.dundee.ac.uk/mhd/pubs.shtml

MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

$$\frac{\mathrm{D}p}{\mathrm{D}t} = -\gamma p \nabla \cdot \mathbf{u} + \frac{(\gamma - 1)}{\mu_0^2 \sigma} |\nabla \times \mathbf{B}|^2$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} \right)$$

Assumptions used

- Fluid
- Isotropic pressure
- LTE
- Ohm's law
- Quasi-neutrality (velocities much less than c)