Compressible MHD

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Outline

- Hydrodynamic Waves and Shocks
- MHD Waves and Shocks in Ideal Single-fluid MHD
- Fast-Mode and Slow-Mode Shocks in the Compression of a Radiating Interstellar Cloud
- Example of Wave-Wave Interactions
- Multifluid Equations for a Weakly Ionised Magnetised Medium
- Wave Damping in a Weakly Ionised Magnetised Medium
- Multifluid Treatments of Perpendicular and Oblique Shocks
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Hydrodynamic Waves and Shocks

Hydrodynamic Equations

$\partial\rho/\partial t + \nabla.(\rho\mathbf{v}) = 0$ (mass conservation)

$\partial(\rho v_i)/\partial t + \sum_j \partial(\rho v_i v_j + \delta_{ij}P)/\partial x_j = f_{oi}$ (force)

$\partial(\frac{1}{2}\rho v^2 + U)/\partial t + \nabla.[(\frac{1}{2}\rho v^2 + U + P)\mathbf{v}] = \varepsilon_o$ *(energy)*

or

$\partial\{\frac{1}{2}\rho v^2 + P/(\gamma - 1)\}/\partial t + \nabla.[\{\frac{1}{2}\rho v^2 + \gamma P/(\gamma - 1)\}\mathbf{v}] = \varepsilon_o$

(energy)

f_{oi} *and* ε_o - i^{th} *momentum component and energy source terms*

Hydrodynamic Sound Waves

$P = K\rho^{\gamma}$, where K is constant. Energy equation if $\varepsilon_o = 0$. Also take $f_{oi} = 0$.

Small perturbations on a uniform static medium:

$$\rho = \rho_0 + \rho_1$$

$$v_1 = v_{11}$$

and

$$P = P_0 + P_1$$

Hydrodynamic Sound Waves - II

$$\partial\rho_1/\partial t + \rho_1\, \partial v_{11}/\partial x_1 = 0$$

and

$$\rho_0\, \partial v_{11}/dt + \partial P_1/\partial x_1 = 0$$

and

$$P_1 = \gamma P_0 \rho_1/\rho_0$$

Hydrodynamic Sound Waves - III

Take

$$\rho_1 \propto v_{11} \propto P_1 \propto exp(i\omega t - ikx_1)$$

Then

$$i\omega\rho_1 - ik\rho_0 v_{11} = 0$$

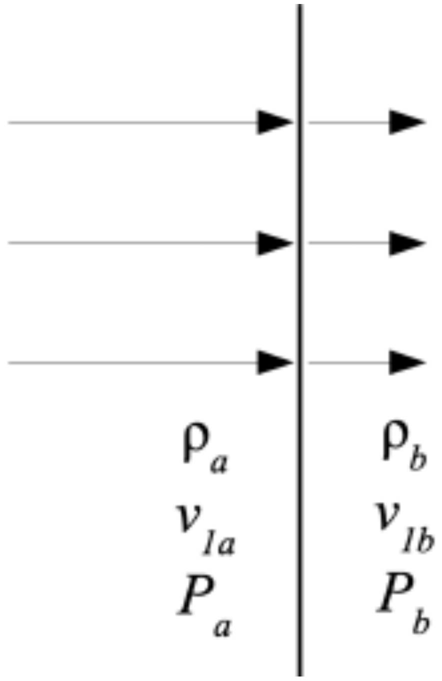
and

$$i\omega\rho_0 v_{11} - ikP_1 = 0$$

Hence,

$$\omega^2 - c_s^2 k^2 = 0 \text{ , with } c_s = (\gamma P_0/\rho_0)^{½}$$

Hydrodynamic Shocks



$$\rho_a v_{1a} = \rho_b v_{1b} \qquad \rho_a v_{1a}^2 + P_a = \rho_b v_{1b}^2 + P_b$$

$$\tfrac{1}{2}\, \rho_a v_{1a}^3 + \{\gamma/(\gamma-1)\}\, P_a v_{1a} = \tfrac{1}{2}\, \rho_b v_{1b}^3 + \{\gamma/(\gamma-1)\}\, P_b v_{1b}$$

Hydrodynamic Shocks - II

Define the Mach number of the shock:

$M_a \equiv v_a / c_{sa}$

Then from the preceding three flux conservation equations:

$\rho_b / \rho_a = v_{1a} / v_{1b} = \{(\gamma + 1)\, M_a^2\} / \{(\gamma - 1)\, M_a^2 + 2\}$

and

$P_b / P_a = \{2\gamma M_a^2 / (\gamma + 1)\} - (\gamma - 1)/(\gamma + 1)$

Hydrodynamic Shocks - III

Strong shocks – $M_a >> 1$

then for $\gamma = 5/3$,

$\rho_b/\rho_a = v_{1a}/v_{1b} \cong 4$

and

$P_b \cong ¾\, \rho_a v_{1a}^2$; $T_b \cong (3/16)\, \mu m_H\, v_{1a}^2/k_B$

Radiative Hydrodynamic Shock

Assume equal upstream and distant downstream temperatures

$$\rho_a v_{1a} = \rho_b v_{1b}$$
$$\rho_a (v_{1a}^2 + c_{sa}^2/\gamma) = \rho_b (v_{1b}^2 + c_{sa}^2/\gamma)$$

For $M_a >> 1$

$$\rho_b = \gamma M_a^2 \rho_a$$

Waves and Shocks in Single-Fluid MHD

Single Fluid MHD Equations

$$\partial\rho/\partial t + \nabla.(\rho\mathbf{v}) = 0$$

$$\partial(\rho v_i)/\partial t + \Sigma_j\, \partial(\rho v_i v_j + \delta_{ij}P)/\partial x_j = (1/\mu_0)[(\nabla \times \mathbf{B}) \times \mathbf{B}]_i + f_{oi}$$

$$\partial\{½\, \rho v^2 + P/(\gamma - 1) + B^2/2\mu_0\}/\partial t +$$
$$\nabla \,.\, [\{½\, \rho v^2 + P\gamma/(\gamma - 1)\}\, \mathbf{v} + (1/\mu_0)\mathbf{B} \times (\mathbf{v} \times \mathbf{B})] = \varepsilon_o$$

$$\partial\mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) + (1/\sigma\mu_0)\, \nabla^2\, \mathbf{B}$$

Waves in Ideal MHD

Small perturbations on a uniform static medium:

$$\rho = \rho_0 + \rho_1$$

$$\boldsymbol{v} = \boldsymbol{v_1}$$

$$P = P_0 + P_1$$

$$\boldsymbol{B} = \boldsymbol{B_0} + \boldsymbol{B_1}$$

$$\rho_1 \propto \boldsymbol{v_1} \propto P_1 \propto \boldsymbol{B_1} \propto exp(i\omega t - i\boldsymbol{k}.\boldsymbol{x})$$

$$\boldsymbol{B_0} = B_0\boldsymbol{e_3}$$

$$\boldsymbol{k} = k_2\boldsymbol{e_2} + k_3\boldsymbol{e_3}$$

Waves in Ideal MHD II

$$[\omega^2 - k_3{}^2 v_A{}^2][\omega^4 - k^2 (c_s{}^2 + v_A{}^2)\,\omega^2 + k^2 k_3{}^2\, c_s{}^2 v_A{}^2] = 0$$

where

$$v_A = B_0/(\mu_0 \rho_0)^{½}$$

is the Alfvén speed

The above dispersion relation has three roots.

Waves in Ideal MHD III Alfvén or Intermediate-Mode Waves

$$\omega^2 = k_3^2\, v_A^2$$

Transverse waves with $\boldsymbol{v_1}$ and $\boldsymbol{B_1}$ parallel to one another but perpendicular to $\boldsymbol{B_0}$ and $\boldsymbol{k}$. Usually limited to circularly polarised waves.

Propagate parallel to $\boldsymbol{B_0}$ and are to first order non-compressive with $\rho_1 = P_1 = 0$.

Some exact solutions to the single-fluid ideal MHD equations correspond to finite amplitude circularly polarised Alfvén waves that are non-compressive.

Waves in Ideal MHD IV Fast-Mode and Slow-Mode Magnetosonic or Magnetoacoustic Waves

$$\omega^2 = \tfrac{1}{2}\, k^2[c_s^{\,2} + v_A^{\,2}]\,[1 \pm (1-\delta)^{1/2}]$$
$$\delta = (4k_3^{\,2}/k^2)\, c_s^{\,2} v_A^{\,2}/(c_s^{\,2} + v_A^{\,2})^2$$

+ sign corresponds to fast-mode; - sign slow-mode

Magnetosonic waves are compressive. For such a wave the component of $\boldsymbol{v_1}$ in the x_1 direction is zero. If $c_s \ll v_A$, for a slow wave the magnitude of the component of $\boldsymbol{v_1}$ parallel to $\boldsymbol{B_0}$ is much larger than the magnitude of the perpendicular component, while for a fast wave the opposite is true.

Waves in Ideal MHD V

$$\omega_f \geq k_3 v_A \geq \omega_s$$

For fast wave for which $\boldsymbol{k}$ is perpendicular to $\boldsymbol{B_0}$:

$$\omega^2 = k^2(c_s{}^2 + v_A{}^2)$$

If $c_s < v_A$ for a slow wave for which $\boldsymbol{k}$ is parallel to $\boldsymbol{B_0}$:

$$\omega^2 = k^2 c_s{}^2$$

Perpendicular Single-Fluid MHD Shocks

$$\rho_a v_{1a} = \rho_b v_{1b}$$

$$\rho_a v_{1a}^2 + P_a + B_a^2/2\mu_0 = \rho_b v_{1b}^2 + P_b + B_b^2/2\mu_0$$

$$\tfrac{1}{2}\rho_a v_{1a}^3 + \{\gamma/(\gamma - 1)\} P_a v_a + B_a^2 v_{1a}/\mu_0 =$$
$$\tfrac{1}{2}\rho_b v_{1b}^3 + \{\gamma/(\gamma-1)\} P_b v_{1b} + B_b^2 v_{1b}/\mu_0$$

$$v_{1a} B_a = v_{1b} B_b$$

Perpendicular Single-Fluid MHD Shocks II

$$2(2 - \gamma)(\rho_b/\rho_a)^2 + [2\gamma(\beta_a + 1) + \beta_a\gamma(\gamma - 1)M_a^{\,2}](\rho_b/\rho_a) - \beta_a\gamma(\gamma+1)\,M_a^{\,2} = 0$$

$\beta_a = 2\mu_0 P_a / B_a^{\,2}$ is the upstream plasma beta

For $\gamma < 2$ only one root is positive. For that root to be greater than unity and relevant to a shock

$$v_{1a}^{\,2} > c_{sa}^{\,2} + v_{Aa}^{\,2}$$

Perpendicular Single-Fluid MHD Shocks III

$$M_{Aa} = v_a (\mu_0 \rho_A)^{½}/B_A$$

is the upstream Alfvénic Mach number

$$B_b/B_a = \rho_b/\rho_a$$

If both the Alfvénic and ordinary sonic Mach numbers are sufficently large, the immediate postshock conditions are given by the ordinary hydrodynamic shock conditions, and the magnetic pressure is limited by the moderate nature of the jump in density that occurs in a non-radiative shock.

Radiative Perpendicular Single-Fluid MHD Shocks

Immediately behind a very strong perpendicular MHD shock the thermal pressure dominates the magnetic pressure.

However, the onset of radiative cooling decreases the thermal pressure and increases the density and magnetic pressure. Sufficient cooling leads the magnetic pressure to become larger than the thermal pressure, limiting further compression of the gas.

Radiative Perpendicular Single-Fluid MHD Shocks II

Strong radiative perpendicular MHD shocks create regions of low plasma beta, in which the thermal pressure is small compared to the magnetic pressure. This is relevant for the formation of magnetically dominated regions in some interstellar clouds.

The density increases across the strong radiative shock by a factor of up to $2^{1/2} M_{Aa}{}^2$. Compare this to $\gamma M_a{}^2$ for a strong isothermal hydrodynamic shock. Creation of low β regions will usually require cooling below the preshock temperature.

Oblique Single-Fluid MHD Shocks

Shock jump conditions for oblique shocks are derived from the full set of flux conservation equations. To classify the solutions for the jump conditions, four velocity zones are identified.

I: $v_1 > v_f$
II: $v_f > v_1 > v_{A1}$
III: $v_{A1} > v_1 > v_s$
IV: $v_s > v_1$

Fast-mode shock I to II
Slow-mode shock III to IV
Intermediate-mode shock I to III OR I to IV OR II to III OR II to IV

Oblique Single-Fluid MHD Shocks II

In an intermediate-mode shock the magnetic field remains in the same plane but its component parallel to the shock front changes sign.

A discontinuity is evolutionary if no small perturbations imposed on the discontinuity surface cause any *instantaneous large* changes in the discontinuity.

Arguments, based on the comparison of the numbers of boundary conditions and outgoing waves at MHD discontinuities, imply that ***intermediate shocks are non-evolutionary.***

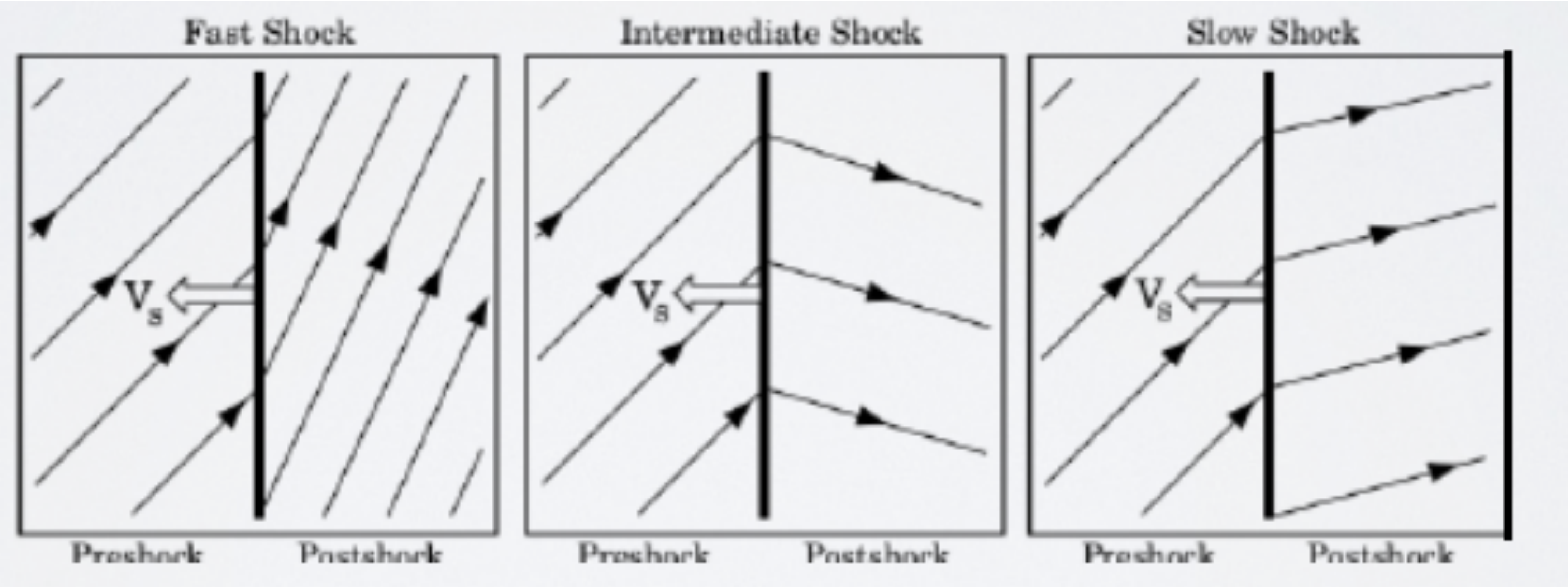
Oblique Single-Fluid MHD Shocks III

A fast-mode shock or a slow-mode shock is compressive and leads to the magnetic field remaining in the same plane and the component of the magnetic field parallel to the shock front maintaining its original sign.

The perpendicular component of the magnetic field is larger behind a fast-mode shock than ahead of it.

In contrast, a slow-mode shock ***reduces the perpendicular component.***

Oblique Single-Fluid MHD Shocks IV



Fast-Mode Shock Followed by a Slow-Mode Shock in a Radiative Initially Warm Interstellar Cloud

Shock – Warm Cloud Simulation

Van Loo et al. (2007, A&A, 471, 213) – 2D axisymmetric (We focus on these results.)
Also Van Loo et al. (2010, MNRAS, 406, 1260) – 3D

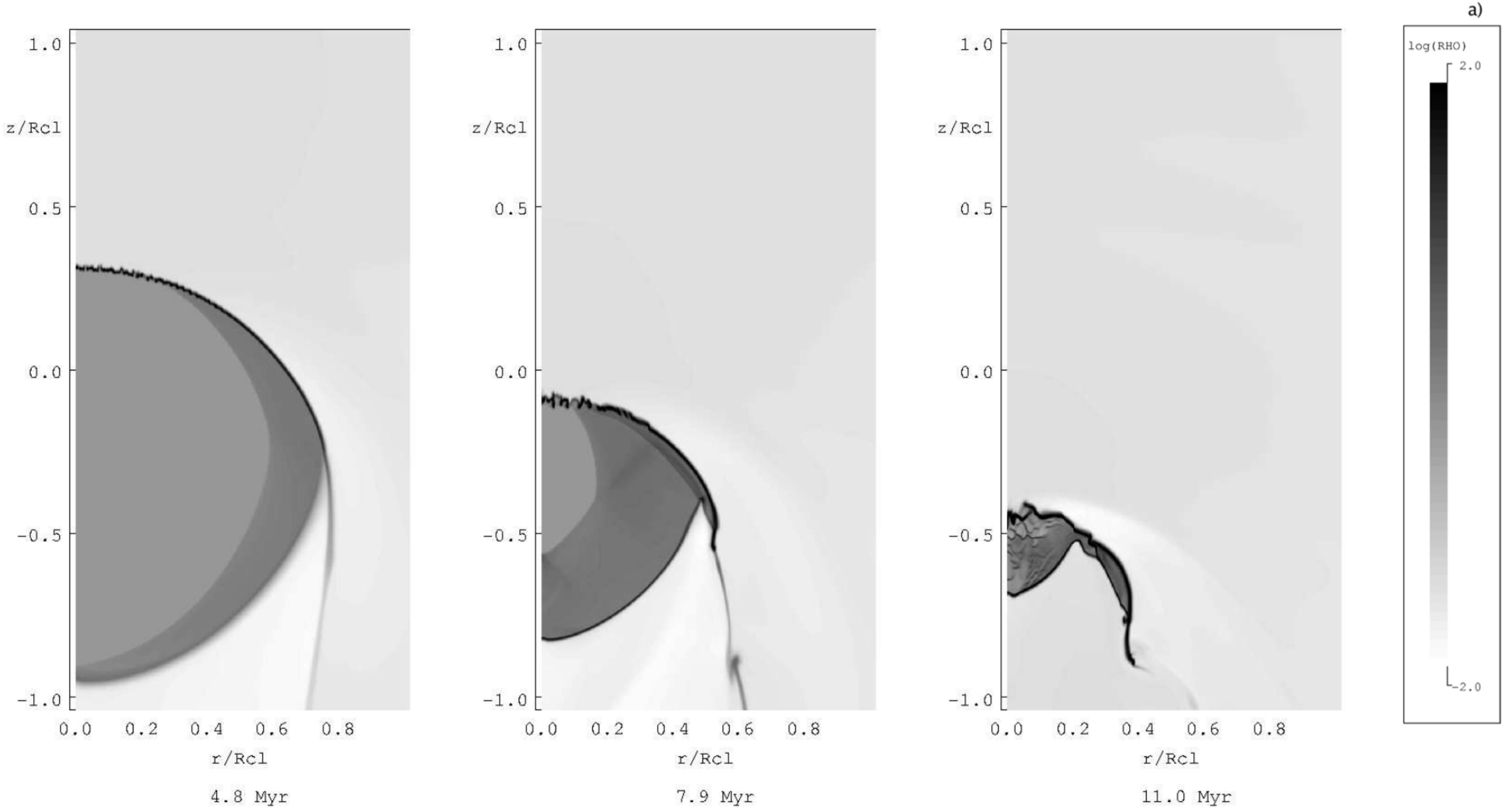
A thermally stable warm (6300 K; 0.45 cm^{-3}; plasma beta of unity) atomic cloud is initially in static equilibrium with the surrounding hot ionised gas (0.01 cm^{-3}). A shock propagating through the hot medium interacts with the cloud.

Shock – Warm Cloud Simulation II

As a fast-mode shock propagates through the cloud, the gas behind it becomes thermally unstable. The β of the gas also becomes much smaller than the initial value of order unity. These conditions are ideal for magnetohydrodynamic waves to produce high-density clumps embedded in a rarefied warm medium.

A slow-mode shock follows the fast-mode shock. Behind this shock a dense shell forms, which subsequently fragments. This is a primary region for the formation of massive stars. The simulations show that only weak and moderate-strength shocks can form cold clouds which have properties typical of giant molecular clouds.

Shock-Cloud Interaction III



An Example of a Wave – Wave Interaction

Fast-Mode Excitation of Slow-Mode Waves

Simulations reported by Falle & Hartquist (2002, MNRAS, 329, 195) showed that in a medium with small plasma beta non-linear steepening of a fast-mode wave with finite but modest amplitude can readily excite the slow-mode as long as the angle between its direction of propagation and the magnetic field is neither too large nor too small. This produces persistent inhomogeneities with a large density contrast.

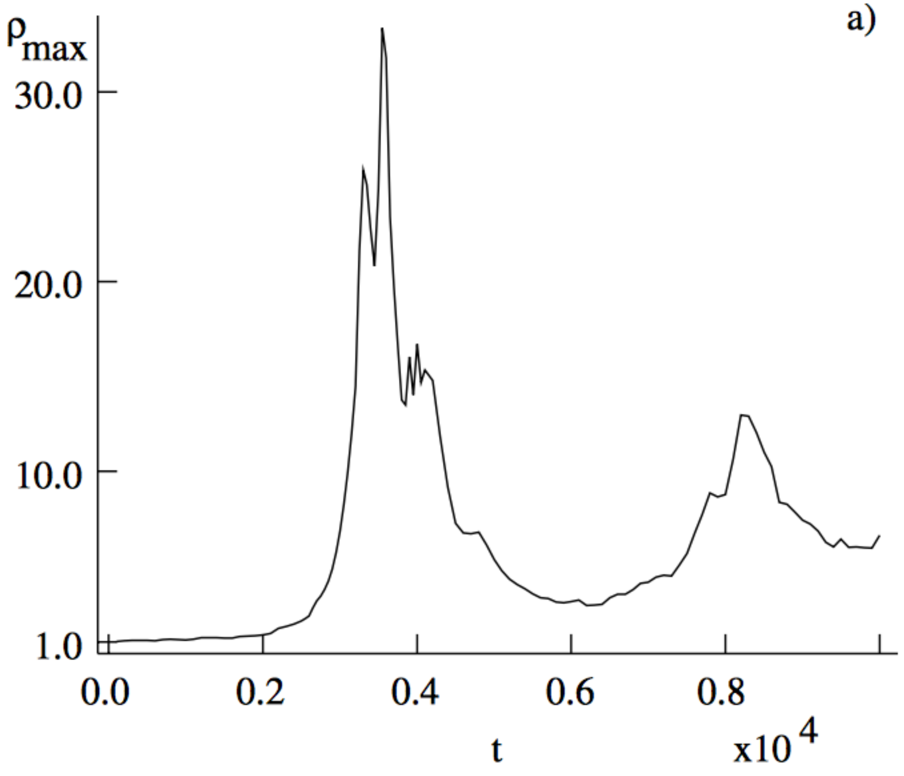
Fast-Mode Excitation of Slow-Mode Waves II

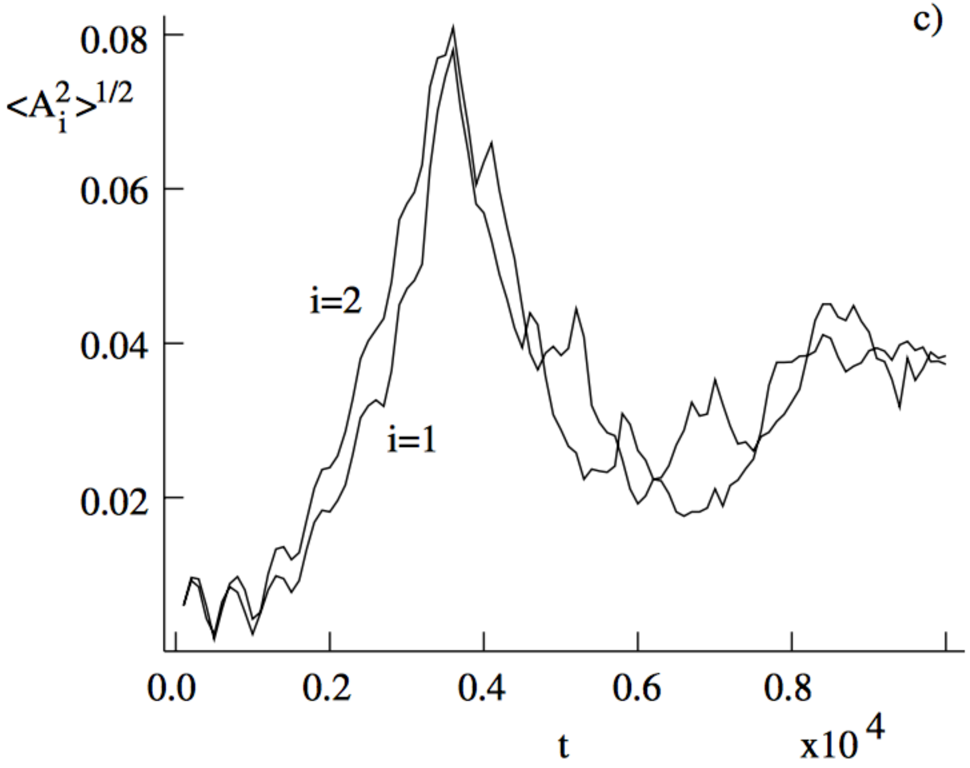
The initial state:
Uniform background with $\rho = 1$, $T = 0.001$, $B_x = 1$, *and* $B_y = \alpha B_x$

Plus fast-mode wave with wavelength of 1000, wavevector in the x direction, and amplitude of 0.1

The ratio of the density perturbation to the velocity perturbation for a slow-mode wave in a plasma with small β is of the order of the ratio of $\beta^{-1/2}$.

Fast-Mode Excitation of Slow-Mode Waves III $\alpha = 0.25$ Results for Density and Slow-Mode Amplitudes





Multifluid Treatments

The Multifluid Equations for Plane-Parallel Flows

The 1-D Multifluid Equations

For the kth fluid

$$\partial \rho_k / \partial t + \partial(\rho_k v_{1k})/\partial x_1 = \sum_{l \neq k} S_{kl}$$

S_{kl} mass transferred per unit time per unit volume from the lth fluid to the kth fluid

$$\partial(\rho_k v_{ik})/\partial t + \partial(\rho_k v_{1k} v_{ik} + P_k)/\partial x_1 =$$
$$n_k q_k(E_i + \sum_{jm} \varepsilon_{ijm} v_{jk} B_m) + \sum_{l \neq k} F_{ikl}$$

F_{ikl} ith component of momentum transferred per unit volume per unit per unit time from the lth fluid to the kth fluid

The 1-D Multifluid Equations II

$$\partial(\rho_k v_k^2/2 + U_k)/\partial t + \partial[(\rho_k v_k^2/2 + U_k + P_k)v_{1k}]/\partial x_1 =$$
$$H_k + \sum_{l\neq k} G_{kl} + n_k q_k \sum_i v_i(E_i + \sum_{jm} \varepsilon_{ijm} v_{jk} B_m)$$

G_{kl} is the rate per unit volume at which energy is transferred from the l^{th} fluid to the k^{th} fluid.
H_k is the rate per unit volume at which the energy density of fluid k changes due to external sources and losses, including those due to radiative cooling.

The 1-D Multifluid Equations III

$$\partial B_1/\partial t = 0\ ; \quad \partial B_2/\partial t = \partial E_3/\partial x_1\ ; \quad \partial B_3/\partial t = -\ \partial E_2/\partial x_1$$

$$\partial B_2/\partial x_1 = \mu_0 J_3\ ; \qquad \partial B_3/\partial x_1 = -\mu_0 J_2$$

$$J_i = \sum_k n_k q_k v_{ik}$$

$$\sum_k n_k q_k = 0$$

Ambipolar Diffusion(Ion-Neutral Streaming) Damping of Waves

Wave Damping

The timescale for a neutral experiencing no forces other than those due to collisions with ions to slow is $1/\alpha_{ni}n_{\mathrm{i}}$ where α_{ni} is the relevant momentum transfer collision rate coefficient and n_i is the number density of ions. The value of α_{ni} is about 2 x 10^{-9} cm^3 s^{-1} if the neutral species is H_2 and the mass of H_2 is significantly less than that of an ion. The analogous slowing down time for ions is $1/\alpha_{in}n_n$ where α_{in} and n_n are the appropriate rate coefficient and the number density of neutrals, respectively. The ratio α_{in}/α_{ni} is equal to the ratio of the mean masses of the neutrals and ions.

Wave Damping II

The Hall parameter β_H is the inverse of the product of the gyrofrequency and the slowing down time due to collisions.

Consider a region where all charged species can be treated as a single fluid and all neutral species as a single fluid. Then an Alfvén wave with a wavevector parallel to the large scale magnetic field will damp due to ion-neutral friction at the rate $v_{Ac}{}^2k^2/2\alpha_{ni}n_i$ if $\omega << \alpha_{ni}n_i$, ω is small compared to the gyrofrequency, the ion's Hall parameter is large and ν and $1/\sigma\mu_0$ are zero. If $\alpha_{ni}n_i\rho_n/\rho_I > \omega > \alpha_{ni}n_i$ such a wave does not propagate.

Wave Damping III

More general investigations of wave dispersion relations than the one cited above, but based on some of the same assumptions, have revealed that in regions with $v_A > c_S$, fast magnetosonic waves, like Alfvén waves, with $\omega > \alpha_{ni}n_i$ are damped rapidly. However, slow magnetosonic waves are not. (Balsara , 1996, ApJ, 465, 775; Mouschovias et al., 2011, MNRAS, 415, 1751)
If $v_A < c_S$, Alfvén and slow magnetosonic waves are damped rapidly at these angular frequencies but fast magnetosonic waves are not.

Wave Damping IV

Lim et al. (2005, MNRAS, 357, 461) included ion-neutral damping in a study of slow-mode excitation by the non-linear steepening of a fast-mode wave.

The initial state: Uniform background with $\rho = 1$, $\beta = 0.00941$, $B_x = 1$, and $B_y = \alpha B_x$; neutral stopping time due to collisions with ions 0.05.

Fast-mode wave with velocity amplitude A propagating in the x direction.

Wave Damping V

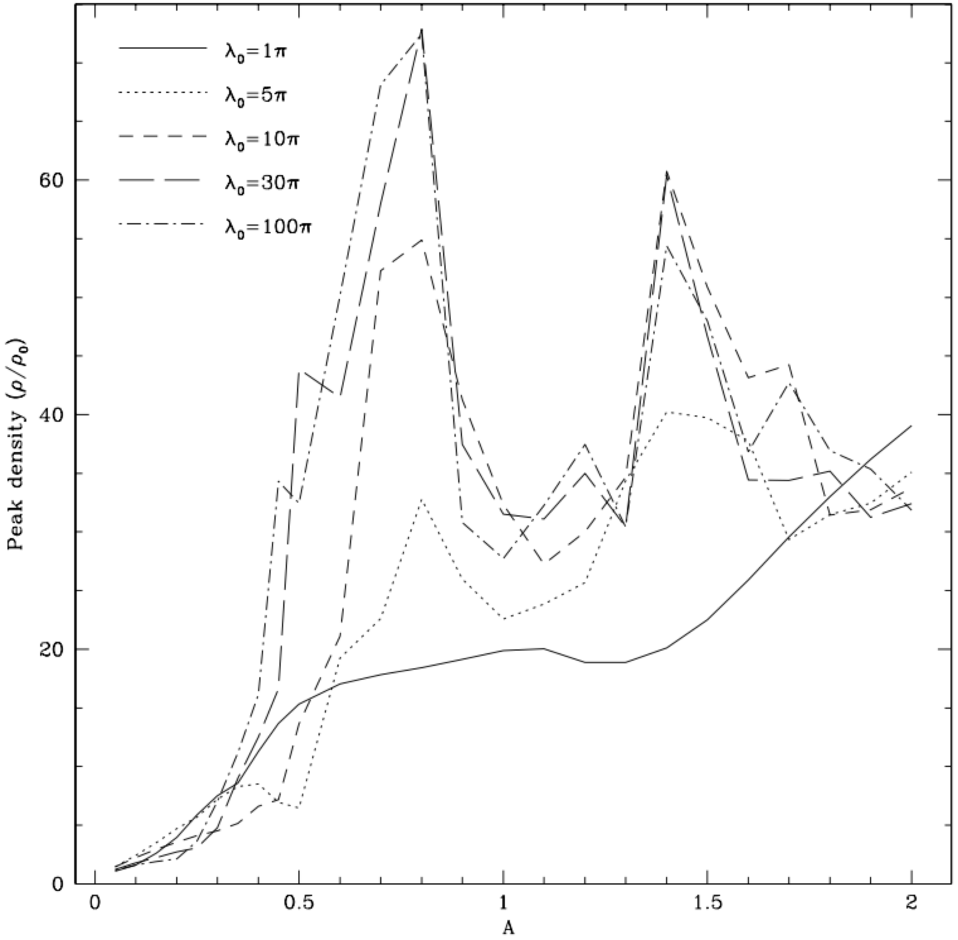


Figure 7. Highest density reached as a function of A and λ_0. $\beta_0 = 9.41 \times 10^{-3}$ and $B_{y0} = 0.25$.

Wave Damping VI

Ion-neutral damping has a significant effect on the evolution when the wavelength of the initial perturbation is up to at least 10π. This is several orders of magnitude larger than the distance that the fast-mode wave will propagate in the slowing down time of a neutral due to collisions with ions and corresponds to about 1 parsec in a dark region of star formation.

Multifluid Models of Shocks

Multifluid Perpendicular Shocks

If $\omega \gg \alpha_{in} n_n$ the wave speed is not significantly affected by the presence of the neutrals and can be well estimated from only the magnetic field strength and direction, the propagation direction, the ion density and the pressure of the ion-electron fluid. Such a wave is a **decoupled wave**.

If $\alpha_{ni} n_i \gg \omega$ the wave speed is significantly affected by the presence of the neutrals and the density and pressure of the neutrals must be included in the calculation of the propagation speed. Such a wave is a **coupled wave**.

Clearly the speed, v_{fc}, of a coupled fast-mode wave is much less than that, v_{fd}, of a decoupled fast-mode wave with the same propagation direction.

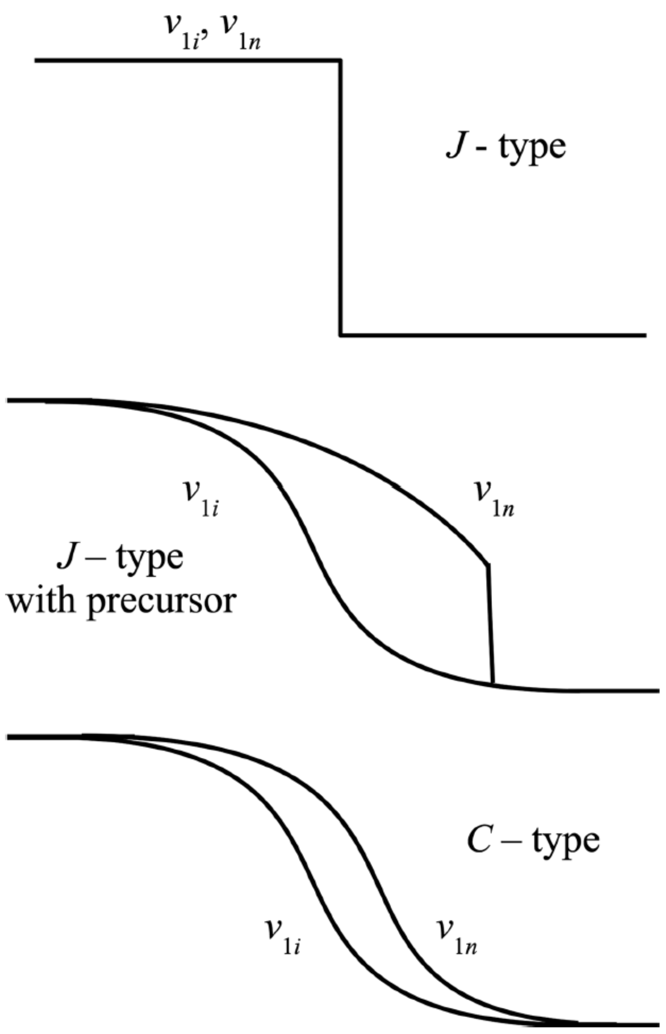
Multifluid Perpendicular Shocks II

The upstream flow to has velocity $v_{1a}\boldsymbol{e_1}$ in the frame of the shock. If $v_{1a} > v_{fd}$, no information can propagate upstream from the shock and it will be a sharp discontinuity. **J-Type**

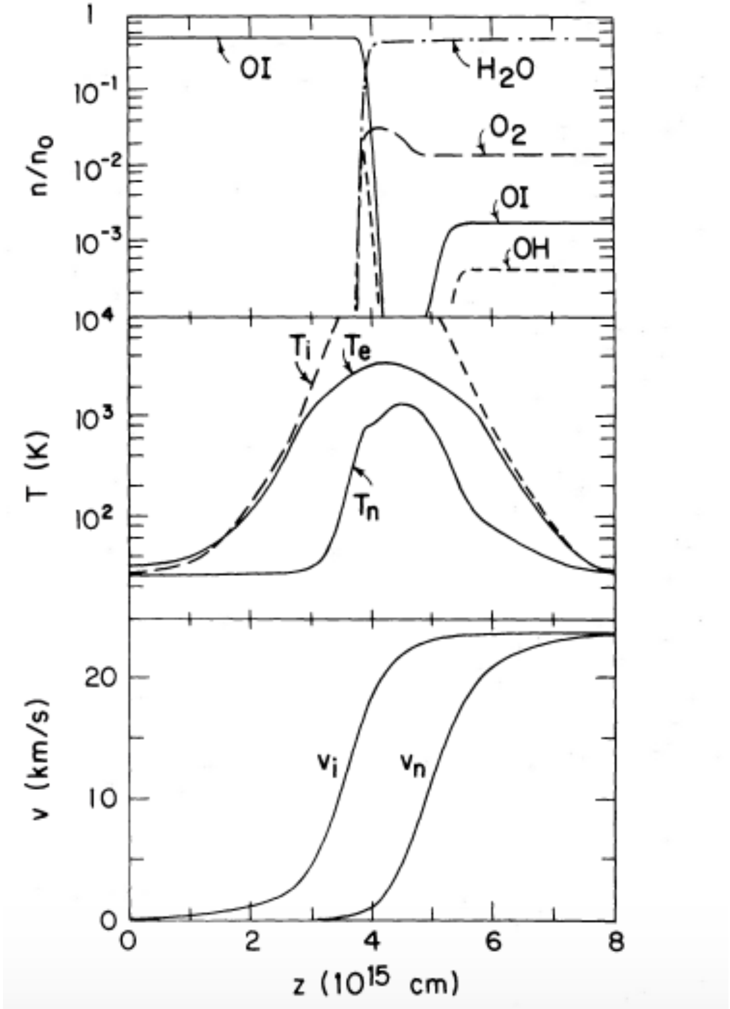
If $v_{fd} > v_{1a} > v_{fc}$ the decoupled waves carry some information upstream. This will cause at least some of the dissipation to occur in a region in which the material decelerates continuously. **J-Type with Precursor**

If radiative cooling in the dissipation zone keeps the temperature in neutral material sufficiently low that the neutral fluid does not undergo a supersonic to subsonic transition in the shock frame, all dissipation will occur in a region of continuous deceleration. **C-Type**

Multifluid Perpendicular Shocks III



Multifluid Perpendicular Shocks IV – Draine et al. (1983, ApJ, 264, 485) $v_{1a} = 25\ km\ s^{-1}$ $n_H = 10^6\ cm^{-3}$ $x_e = 10^{-8}$ $B_a = 10^{-3}\ G$



Multifluid Perpendicular Shocks V Dust Grains

$n_d \pi a^2 / n_H = 10^{-21}\ cm^2$

If dust is well coupled to the magnetic field, then it provides more drag on the neutrals than the ions do, if $x_i < 5 \times 10^{-7} (v_i - v_n)/10\ km\ s^{-1}$

If $\beta_H >> 1$ or if grains carry a significantly larger fraction of negative charge than electrons do, then dust is well-coupled to the magnetic field. However, these conditions are not always met. Furthermore, electron and ion collisions with dust grains significant affect the fractional ionisation.

Multifluid Perpendicular Shocks VI Dust Grains Part 2

Four fluid (neutrals, ions, electrons, dust) models (Pilipp et al. 1990, MNRAS, 243, 685) in which the charges on dust and the gas phase ion and electron abundances were calculated self-consistently and grain dynamics were treated on an equal footing with the dynamics of the other fluids showed notable deviations from the the models of Draine et al. (1983) for preshock densities of about 10^7 cm^{-3} and above.

Multifluid Oblique Shocks

Pilipp & Hartquist (1994, MNRAS, 267, 801) applied the four fluid approach of Pilipp et al. (1990) to oblique shocks. Assuming that the shocks were steady and integrating in the downstream direction, they were able to find only intermediate-mode shock solutions.

Multifluid Oblique Shocks II

Wardle (1998, MNRAS, 298, 507) showed that this is due to the saddle-point nature of the downstream conditions. He overcame this problem by integrating the equations in the upstream direction and found solutions for fast-mode shocks in which field rotation occurs. He showed that the inclusion of field and velocity components that do not lie in the plane of the shock propagation direction and the upstream magnetic field leads to the dissipation regions in the model shocks being narrower than they would be otherwise.

Multifluid Oblique Shocks III

Wardle’s approach does not allow the inclusion of non-equilibrium microphysics and chemistry. Falle (2003, MNRAS, 344, 1210) developed a time-dependent approach that did.

Applications of Falle’s method have included one to the sputtering of grains in shocks driven into dusty molecular regions by the outflows of recently formed stars (Van Loo et al. 2013, MNRAS, 428, 381).

Multifluid Oblique Shocks IV

The results of Van Loo et al. (2013) indicate that near the sputtering threshold shock speed the sputtering depends significantly on the angle θ between the shock propagation direction and the upstream magnetic field. For example, a comparison of results for shocks with speeds of 25 km s^{-1} and propagating into upstream media with hydrogen nuclei number densities of 10^5 cm^{-3} shows that about two orders of magnitude more elemental silicon is injected into the gas phase when $\theta = \pi/6$ than when $\theta = \pi/3$.