Simulation of particle systems interacting through hitting times

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Introduction

- We propose and analyse numerical schemes for the simulation of a specific McKean–Vlasov equation, where the interaction derives from feedback on the system when a certain threshold is hit.

$$Y_t = Y_0 + W_t - \alpha L_t, \qquad t \in [0, T], \tag{1}$$

$$L_t = \mathbb{P}(\tau \le t), \qquad t \in [0, T], \tag{2}$$

$$\tau = \inf\{t \in [0, T] : Y_t \le 0\}, \tag{3}$$

where $\alpha, T \in \mathbb{R}_+$, W is a standard Brownian motion and Y_0 is $\mathbb{R}_+$-valued random variable, independent of W.

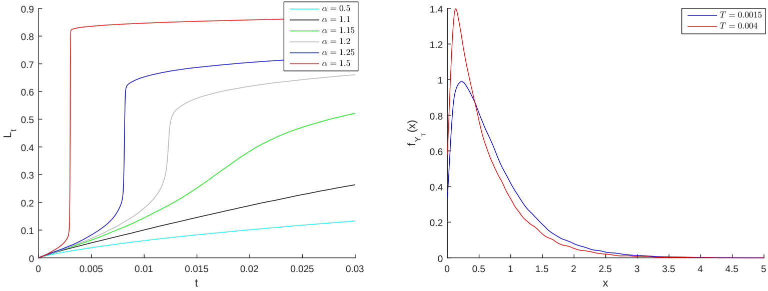
Introduction

- One motivation for studying these equations comes from mathematical finance, in particular, systemic risk.
- A large interconnected banking network can be approximated by a particle system with interactions by which the default of one firm, modeled as the hitting of a lower default threshold of its value, causes a downward move in the firm value of others.
- More details can be found in [Hambly et al., 2018] and [Nadtochiy and Shkolnikov, 2017].
- This model can also be viewed as the large pool limit of a structural default model for a pool of firms where interconnectivity is caused by mutual liabilities, such as in [Lipton, 2016]. The limit results can be found in [Lipton et al., 2018].

Introduction

- Theoretical properties of (1)–(3) have been studied in [Hambly et al., 2018], who prove the existence of a differentiable solution $(L_t)_{0<t<t_*}$ up to an "explosion time" t_*.
- Conversely, they show that L cannot be continuous for all t for α above a threshold determined by the law of Y_0. Such systemic events where discontinuities occur are also referred to as "blow-ups" in the literature.
- The question of the constructive solution, however, remained open.

The left plot shows the formation of a discontinuity in the loss function $t \to L_t$ for increasing α, with $Y_0 \sim \mathrm{Gamma}(1.5, 0.5)$. The density of Y_T for T before and after the shock is displayed in the right panel.



(a) (b)

Figure: (a) L_t for different α near the jump (b) Distribution of Y_T for $Y_T > 0$ before and after the jump.

Discrete time Monte Carlo scheme for simulation of the loss process

Require: N — number of Monte Carlo paths
Require: n — number of time steps: $0 < t_1 < t_2 < \ldots < t_n$

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Draw N samples of Y_0 (from initial distribution) and W (a Brownian path)
Define \hat{L}_0 = 0
for i = 1 : n do
    Estimate \tilde{L}_{t_i} by \hat{L}^N_{t_i} = \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{\min_{j<i} \hat{Y}^{(k)}_{t_j} \le 0\}}
    for k = 1 : N do
        Update \hat{Y}^{(k)}_{t_i} = Y^{(k)}_0 + W^{(k)}_{t_i} - \alpha \hat{L}^N_{t_i}
    end for
end for
```

Draw N samples of Y_0 (from initial distribution) and W (a Brownian path)

Define $\hat{L}_0 = 0$

for $i = 1 : n$ **do**

  Estimate $\tilde{L}_{t_i}$ by $\hat{L}^N_{t_i} = \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{\min_{j<i} \hat{Y}^{(k)}_{t_j} \le 0\}}$

  for $k = 1 : N$ **do**

    Update $\hat{Y}^{(k)}_{t_i} = Y^{(k)}_0 + W^{(k)}_{t_i} - \alpha \hat{L}^N_{t_i}$

  end for

end for

Assumptions and convergence results

- Hölder continuity at 0 of the initial density, is key for the regularity of the solution. The Hölder exponent will also limit the rate of convergence of the discrete time schemes.

Assumption

We assume that Y_0 *has a density* f_{Y_0} *supported on* $\mathbb{R}_+$ *such that*

$$f_{Y_0}(x) \leq Bx^{\beta}, \qquad x \geq 0 \tag{4}$$

for some $\beta \in (0, 1]$.

Assumptions and convergence results

- Under Assumption 1, we can refer to Theorem 1.8 in [Hambly et al., 2018] for the existence of a unique, differentiable solution $t \to L_t$ for (1)–(3) up to time

$$t_* := \sup\{t > 0 : ||L||_{H^1(0,t)} < \infty\} \in [0, \infty],$$

and a corresponding $\hat{B}$ such that for every $t < t_*$

$$L'_t \leq \hat{B} t^{-\frac{1-\beta}{2}} \ a.e. \tag{5}$$

- Integrating (5), we have for future reference a bound for L_t,

$$L_t \leq \tilde{B} t^{\frac{1+\beta}{2}}, \tag{6}$$

where $\tilde{B} = 2\hat{B}/(1+\beta)$.

Assumptions and convergence results

- The following assumption will be used to control the propagation of the discretisation error, by bounding the density (especially at 0) of the running minimum of Y and its approximations.

Assumption

We assume that $T < \min(T^*, t_*)$, *where* T^* *is defined by*

$$\alpha B \left[\sqrt{\frac{2T^*}{\pi}} + \alpha \tilde{B} (T^*)^{\frac{1+\beta}{2}} \right]^{\beta} = 1, \qquad (7)$$

with B *and* $\tilde{B}$ *the smallest constants such that* (4) *and* (6) *hold for given* β.

Assumptions and convergence results

Consider a uniform time mesh $0 = t_0 < t_1 < \ldots < t_n = T$, where $t_i - t_{i-1} = h$, and a discretized process, for $1 \leq i \leq n$,

$$\begin{aligned} \tilde{Y}_{t_i} &= Y_0 + W_{t_i} - \alpha \tilde{L}_{t_i}, & (8) \\ \tilde{L}_{t_i} &= \mathbb{P}(\tilde{\tau} < t_i), & (9) \\ \tilde{\tau} &= \min_{0 \leq j \leq n} \{\tilde{Y}_{t_j} \leq 0\}. & (10) \end{aligned}$$

We extend $\tilde{L}_{t_i}$ to $[0, T]$ by setting $\tilde{L}_s = \tilde{L}_{t_{i-1}}$ for $t_{i-1} < s < t_i$.

Assumptions and convergence results: main results

Theorem (1)

Consider $\tilde{L}_{t_i}$ from (8)–(10) and L_t from (1)–(3). Then, for any $\delta > 0$, there exists $C > 0$ independent of h such that

$$\max_{i \leq n} |\tilde{L}_{t_i} - L_{t_i}| \leq C h^{\frac{1}{2} - \delta}. \tag{11}$$

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Theorem (2)

For all $i \leq n$,

$$\hat{L}_{t_i} \xrightarrow[N \to \infty]{\mathbb{P}} \tilde{L}_{t_i}. \tag{12}$$

-

Proof of Theorem 1

- We split the error into two contributions

$$|\tilde{L}_{t_i} - L_{t_i}| = \left| \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \right|$$
$$\leq \left| \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \right|$$
$$+ \left| \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \right|.$$

- The second term can be estimated as

$$0 \leq \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \leq \gamma h^{\frac{1}{2}-\delta},$$

for any $\delta > 0$.

Proof of Theorem 1

- We split the error into two contributions

$$|\tilde{L}_{t_i} - L_{t_i}| = \left| \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \right|$$
$$\leq \left| \mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \right|$$
$$+ \left| \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \right|.$$

- The second term can be estimated as

$$0 \leq \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) - \mathbb{P}\left(\inf_{s<t_i} Y_s > 0\right) \leq \gamma h^{\frac{1}{2}-\delta},$$

for any $\delta > 0$.

Proof of Theorem 1

- Now we shall proceed by induction to estimate $|\tilde{L}_{t_i} - L_{t_i}|$. For $t_0 = 0$, we have $L_0 = \tilde{L}_0$. Assume we have shown $\tilde{L}_{t_j} = L_{t_j} - \tilde{C}_j h^{\frac{1}{2}-\delta}$ for $j < i$, where $\tilde{C}_j \geq 0$ as $\tilde{L}_{t_j} \leq L_{t_j}$. Then,
- The first term can be estimated as

$$\mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right)$$
$$= \mathbb{P}\left(\min_{j<i}\left(Y_{t_j} + \alpha \tilde{C}_j h^{\frac{1}{2}-\delta}\right) > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right)$$
$$\leq \mathbb{P}\left(\min_{j<i} Y_{t_j} > -\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right)$$
$$= \bar{F}_i(0) - \bar{F}_i\left(-\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right)$$
$$\leq \alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta} \sup_{\theta\in[0,1]} \bar{\varphi}_i\left(-\theta\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right)$$

- Now we shall proceed by induction to estimate $|\tilde{L}_{t_i} - L_{t_i}|$. For $t_0 = 0$, we have $L_0 = \tilde{L}_0$. Assume we have shown $\tilde{L}_{t_j} = L_{t_j} - \tilde{C}_j h^{\frac{1}{2}-\delta}$ for $j < i$, where $\tilde{C}_j \geq 0$ as $\tilde{L}_{t_j} \leq L_{t_j}$. Then,
- The first term can be estimated as

$$
\begin{aligned}
&\mathbb{P}\left(\min_{j<i} \tilde{Y}_{t_j} > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \\
&= \mathbb{P}\left(\min_{j<i} \left(Y_{t_j} + \alpha \tilde{C}_j h^{\frac{1}{2}-\delta}\right) > 0\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \\
&\leq \mathbb{P}\left(\min_{j<i} Y_{t_j} > -\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right) - \mathbb{P}\left(\min_{j<i} Y_{t_j} > 0\right) \\
&= \bar{F}_i(0) - \bar{F}_i\left(-\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right) \\
&\leq \alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta} \sup_{\theta \in [0,1]} \bar{\varphi}_i\left(-\theta\alpha \max_{j<i} \tilde{C}_j h^{\frac{1}{2}-\delta}\right)
\end{aligned}
$$

Proof of Theorem 1

- We got an estimate for $\bar{\varphi}_i$

$$\bar{\varphi}_i\left(-\theta\frac{\alpha}{2}\max_{j<i}\tilde{C}_j h^{\frac{1}{2}-\delta}\right)\leq B\left[\sqrt{\frac{2t_i}{\pi}}+\alpha\tilde{B}t_i^{\frac{1+\beta}{2}}\right]^{\beta},$$

- As a result, we have the following inequality for $\tilde{C}_i$,

$$\tilde{C}_i\leq\alpha\max_{j<i}\tilde{C}_j B\left[\sqrt{\frac{2t_i}{\pi}}+\alpha\tilde{B}t_i^{\frac{1+\beta}{2}}\right]^{\beta}+\gamma,$$

- Hence $\tilde{C}_i$ is bounded independent of i and h by Assumption 2.
- By induction we get (3).
- We also proved that the result of Theorem 1 can be extended up to the explosion time t_* under certain conditions on the parameters.

Main results

Theorem

Consider $\tilde{L}_{t_i}$ *from* (8)–(10) *and* L_t *from* (1)–(3). *Then, for any* $\delta > 0$, *there exists* $C > 0$ *independent of* h *such that*

$$\max_{i \leq n} |\tilde{L}_{t_i} - L_{t_i}| \leq C h^{\frac{1}{2} - \delta}.$$

-

Theorem

For all $i \leq n$,

$$\hat{L}_{t_i} \xrightarrow[N \to \infty]{\mathbb{P}} \tilde{L}_{t_i}.$$

-

Proof of Theorem 2

- We prove the convergence in probability of

$$\hat{L}_{t_i} = \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{\{\min_{j<i} \hat{Y}_{t_j}^{(k)} \leq 0\}}$$

  in Algorithm 1 to $\tilde{L}_{t_i}$ as $N \to \infty$.
- We note that we cannot directly apply the law of large numbers, as the summands are dependent through $\hat{L}_{t_j}^N, j < i$.
- However, we show that the dependence diminishes (i.e., the covariance goes to zero) as $N \to \infty$, which easily gives convergence, albeit without a Central Limit Theorem-type error estimate or a rate for the variance.

Proof of Theorem 2

- First, we formulate an auxiliary lemma.

Lemma

Consider $i \leq n$. Assume for all $j < i$

$$\hat{L}^N_{t_j} \xrightarrow{\mathbb{P}} \tilde{L}_{t_j}.$$

Then,

$$\mathbb{E}[\hat{L}^N_{t_i}] \xrightarrow[N\to\infty]{} \tilde{L}_{t_i}$$
$$\mathbb{V}[\hat{L}^N_{t_i}] \xrightarrow[N\to\infty]{} 0.$$

Proof of Theorem 2

- The proof is immediate by induction. The statement is true for $i = 0$. Now take $i \geq 1$.
- By the Lemma, there exists N^* such that for all $N > N^*$,

$$|\mathbb{E}[\hat{L}_{t_i}^N] - \tilde{L}_{t_i}| \leq \frac{\varepsilon}{2}.$$

- Thus, by Chebyshev's inequality, we have

$$\mathbb{P}(|\hat{L}_{t_i}^N - \tilde{L}_{t_i}| > \varepsilon) \leq \mathbb{P}\left(|\hat{L}_{t_i}^N - \mathbb{E}[\hat{L}_{t_i}^N]| > \frac{\varepsilon}{2}\right) \leq \frac{4\mathbb{V}[\hat{L}_{t_i}^N]}{\varepsilon^2}.$$

- Using again the Lemma, we have that $\mathbb{V}[\hat{L}_{t_i}^N] \xrightarrow[N\to\infty]{} 0$. Hence,

$$\hat{L}_{t_i}^N \xrightarrow[N\to\infty]{\mathbb{P}} \tilde{L}_{t_i},$$

for i and by induction we have proved the theorem.

Convergence improvement: Brownian bridges

- Next, we improve our scheme by using a Brownian bridge strategy to estimate the hitting probabilities.
- This is very similar to barrier option pricing.
- In order to do this, we consider the process

$$\begin{aligned} \check{Y}_t &= Y_0 + W_t - \alpha \check{L}_t, \qquad t \in [t_i, t_{i+1}), && (13) \\ \check{L}_t &= \mathbb{P}(\check{\tau} < t_i), \qquad t \in [t_i, t_{i+1}), && (14) \\ \check{\tau} &= \inf_{0 \le s \le T} \{\check{Y}_s \le 0\}. && (15) \end{aligned}$$

- Then, for each Brownian path $(W_t^{(k)})_{t \ge 0}$, we compute $\bar{Y}_t^{(k)} = Y_0^{(k)} + W_t^{(k)} - \alpha \bar{L}_t^N$ in (t_i, t_{i+1}), where $\bar{L}_t^N$ is an N-sample estimator of $\check{L}_{t_i}$ given on the next slide.

Convergence improvement: Brownian bridges

- Hence, using Brownian bridges, we compute

$$\begin{aligned} p_{t_i}^{(k)} &= \mathbb{P}\left(\inf_{s<t_i} \bar{Y}_s^{(k)} > 0 | \bar{Y}_0^{(k)}, \ldots, \bar{Y}_{t_i}^{(k)}\right) \\ &= \prod_{j=1}^{i} \mathbb{P}\left(\inf_{s\in[t_{j-1},t_j)} \bar{Y}_s^{(k)} > 0 \mid \bar{Y}_{t_{j-1}}^{(k)}, \bar{Y}_{t_j}^{(k)}\right) \\ &= \prod_{j=1}^{i} \left(1 - \exp\left(-\frac{2(\bar{Y}_{t_{j-1}}^{(k)} \vee 0)(\bar{Y}_{t_j-}^{(k)} \vee 0)}{h}\right)\right). \end{aligned}$$

Thus, a natural choice for $\bar{L}_{t_i}^N$ is

$$\bar{L}_{t_i}^N = \frac{1}{N} \sum_{k=1}^{N} \left(1 - p_{t_i}^{(k)}\right). \tag{16}$$

Discrete time scheme for simulation of the loss process using Brownian bridges

As a result, the new algorithm with the Brownian bridge modification is the following

Require: N — number of Monte Carlo paths
Require: n — number of time steps: $0 < t_1 < t_2 < \ldots < t_n$

```
Draw N samples Y_0 (from the initial distribution) and W (a Brownian path)
for i = 1 : n do
    Estimate \bar{L}^N_{t_i} = \frac{1}{N} \sum_{k=1}^{N} (1 - p^{(k)}_{t_i})
    for k = 1 : N do
        Update \bar{Y}^{(k)}_{t_i} = Y^{(k)}_0 + W^{(k)}_{t_i} - \alpha \bar{L}^N_{t_i}
    end for
end for
```

Main results: Modification using Brownian bridges

- The convergence rate for (14) is given in the following theorem

Theorem

Consider $\breve{L}_t$ from (13)–(15) and L_t from (1)–(3), $\beta \in (0,1]$ from Assumption 1. Then, there exists $C > 0$ independent of h such that

$$\max_{i \leq n} |\breve{L}_{t_i} - L_{t_i}| \leq C h^{\frac{1+\beta}{2}}.$$

- The convergence rate can be improved to 1 using a non-uniform grid

Corollary

Consider a non-uniform time mesh $t_i = (ih)^{\frac{2}{1+\beta}}$ for $0 \leq i \leq n$ with $h = T^{\frac{1+\beta}{2}}/n$. Then, there exists $C_1 > 0$, independent of h, such that

$$\max_{i \leq n} |\breve{L}_{t_i} - L_{t_i}| \leq C_1 h.$$

- We shall proceed by induction. Assume we have shown that $|\breve{L}_{t_j} - L_{t_j}| \leq C_j h^{\frac{1+\beta}{2}}$ for all $j < i$ with some $C_j > 0$, and we want to estimate $|\breve{L}_{t_i} - L_{t_i}|$.
- First, we have

$$\sup_{t_j \leq s < t_{j+1}} |\breve{L}_s - L_s| \leq |\breve{L}_{t_j} - L_{t_j}| + \hat{B} h^{\frac{1+\beta}{2}} \leq (C_j + \hat{B}) h^{\frac{1+\beta}{2}},$$

since $L'_\zeta \leq \hat{B} \zeta^{-\frac{1-\beta}{2}}$.

- Now consider

$$\begin{aligned}
|\breve{L}_{t_i} - L_{t_i}| &= \left| \mathbb{P}\left(\inf_{s \leq t_i} \breve{Y}_s > 0 \right) - \mathbb{P}\left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&= \left| \mathbb{P}\left(\inf_{s \leq t_i} (Y_s + \alpha(L_s - \breve{L}_s)) > 0 \right) - \mathbb{P}\left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&\leq \left| \mathbb{P}\left(\inf_{s \leq t_i} Y_s > -\alpha \sup_{s < t_i} |\breve{L}_s - L_s| \right) - \mathbb{P}\left(\inf_{s \leq t_i} Y_s > 0 \right) \right| \\
&\leq \mathbb{P}\left(\inf_{s \leq t_i} Y_s > -\alpha \max_{j<i} (C_j + \hat{B}) h^{\frac{1+\beta}{2}} \right) - \mathbb{P}\left(\inf_{s \leq t_i} Y_s > 0 \right) \\
&\leq \alpha \max_{j<i} (C_j + \hat{B}) \sup_{\theta \in [0,1]} \varphi\left(-\theta\alpha \max_{j<i} (C_j + \hat{B}) h^{\frac{1+\beta}{2}} \right) h^{\frac{1+\beta}{2}},
\end{aligned}$$

where $\varphi_i(x)$ is the density of $\inf_{s<t_i} Y_s$.

- Then, using an estimate for the density, we have

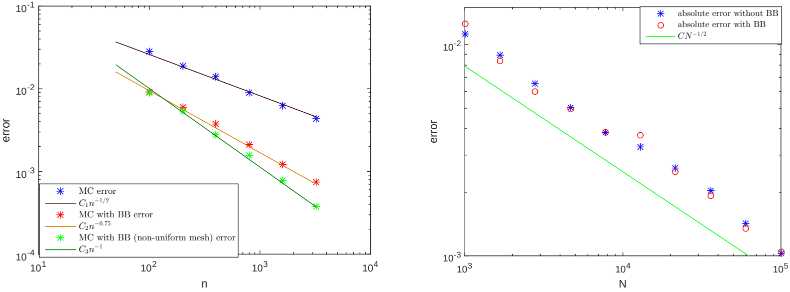
$$C_i \leq \alpha B \max_{j<i}(C_j + \hat{B}) \left[\sqrt{\frac{2t_i}{\pi}} + \alpha \tilde{B} t_i^{\frac{1+\beta}{2}} \right]^\beta$$

$$\leq \gamma \sum_{k=0}^{i} (\alpha B)^k \prod_{j=1}^{k} \left[\sqrt{\frac{2t_j}{\pi}} + \alpha \tilde{B} t_j^{\frac{1+\beta}{2}} \right]^\beta ,$$

where $\gamma = \alpha B \hat{B} \left[\sqrt{\frac{2T}{\pi}} + \alpha \tilde{B} T^{\frac{1+\beta}{2}} \right]^\beta$.

- Thus, C_i is bounded independent of h and i by (7).
- By induction we get the proof.

Numerical results



(a) (b)

Figure: Error of the loss process at $t = T$ for $Y_0 \sim \text{Gamma}(3/2, 1/2)$: (a) for increasing number n of timesteps; (b) for increasing number N of samples, both for Algorithms 1 and 2.

Numerical results

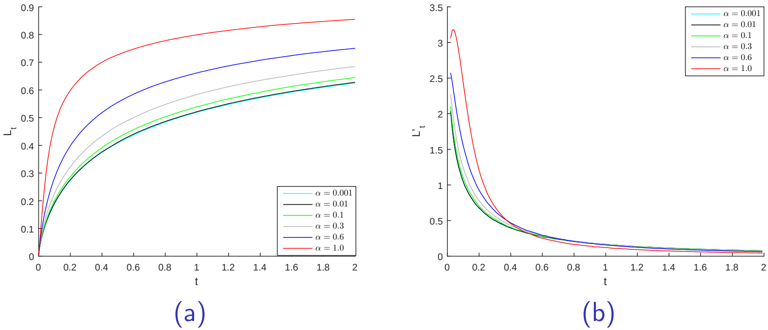


Figure: L_t and L'_t for different values of α.

Conclusion

- We have developed particle methods with explicit timestepping for the simulation of (1)-(3).
- Convergence with a rate up to 1 in the timestep is shown under a condition on the model parameters and time horizon, when the loss function is differentiable.
- This opens up several theoretical and practical questions. The efficiency of the method could be significantly improved by a simple application of multilevel simulation.
- Theoretically, one would like guaranteed convergence also in the blow-up regime. This requires the choice of an appropriate metric – the Skorokhod distance may be suitable.
- Lastly, it would be interesting to investigate the extension to the models in [Nadtochiy and Shkolnikov, 2017] and [Delarue et al., 2015] in more detail.

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