A Mean Field Game Of Optimal Portfolio Liquidation.

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BSDEs, Information and McKean-Vlasov equations

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Outline

Outline

Unwinding large positions is part of day-to-day business.

... of banks, funds, insurance companies, energy companies, ...

- Sell x shares of $\cdots$ within T minutes using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
		Bid/Ask Orders											
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

- Limited market liquidity leads to a price impact.
- Aim: Optimize trading strategies to minimize execution costs.

Price impact modelling.

- Fix an initial position $x \in \mathbb{R}$ and a time horizon T.
- Execution strategy X: finite variation process satisfying $X_{0^-} = x$ and $X_{T^+} = 0$.
- There is an unaffected price process S^0. To disentangle investment from execution strategies, one often assumes that S^0 is a martingale.
- A price impact model assigns to each execution strategy X a realized price process S^X.
- Typically: $S^X \geq S^0$ if X is a pure buying strategy and $S^X \leq S^0$ if X is a pure selling strategy

Continuous-time Almgren & Chriss model (2000).

- Execution strategies have absolutely continuous paths:

$$X_t = x - \int_0^t \xi_s ds.$$

- Price impact consists of two components

$$S_t^X = S_t^0 + \underbrace{\int_0^t g(\xi_s) ds}_{\text{permanent}} + \underbrace{h(\xi_t)}_{\text{temporary}} .$$

- Gatheral (2010): Take $g(x) = -\kappa x$ to rule out price manipulation.

Expected Revenues.

Assume

$$S_t^X = S_t^0 - \int_0^t \kappa_s \xi_s ds - \eta_t \xi_t.$$

Revenues obtained from following X (with $X_T = 0$)

$$R_T(X) = -\int_0^T S_t^X dX_t.$$

Integrating by parts $\rightsquigarrow$ decomposition of expected revenues

$$\mathbb{E}[R_T(X)] = \underbrace{xS_0^0}_{\text{naive book value}} - \underbrace{\mathbb{E}\left[\int_0^T \kappa_s \xi_s X_s ds\right]}_{\text{costs entailed by perm impact}} - \underbrace{\mathbb{E}\left[\int_0^T \eta_s (\xi_s)^2 ds\right]}_{\text{costs entailed by temp impact}}$$

(Non exhaustive) literature review.

- Mean-variance optimization: Almgren & Chriss (1999, 2000), Almgren (2003), Lorenz & Almgren (2011), ...
- Expected-Utility maximization: Schied & Schöneborn (2009), Schied, Schöneborn & Tehranchi (2010), Schöneborn (2011), ...
- Time-averaged Risk Measures: Gatheral & Schied (2011), Forsyth, Kennedy, Tse & Windcliff (2012), Ankirchner & Kruse (2012), ...

- Overview : Guéant (2016): The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making.

Extensions

- Including a dark pool.
- Models with transient impact.
- Models with non aggresive strategies.
- ...

Linear quadratic control problem.

Admissible controls: $\xi \in \mathcal{A}(t,x)$ iff

$$X_s = x - \int_t^s \xi_u du, \ s \in [t,T]$$

with the terminal state constraint: $X_T = 0$

Cost parameters η, λ and κ: non negative and random.

- Expected running execution costs

$$\mathcal{J}(t,\xi) = \mathbb{E}\left[\int_t^T \left(\eta_s (\xi_s)^2 + \kappa_s \xi_s X_s + \underbrace{\lambda_s (X_s)^2}_{\text{risk aversion}} \right) ds \middle| \mathcal{F}_t \right]$$

- Value function

$$v(t,x) = \inf_{\xi \in \mathcal{A}(t,x)} \mathcal{J}(t,\xi)$$

Related literature.

- Penalization and monotone convergence argument.
  - A.P. (2006).
  - S. Ankirchner, M. Jeanblanc & T. Kruse (2013).
  - P. Graewe, U. Horst & J. Qiu (2015).
  - T. Kruse & A.P. (2016).
  - S. Ankirchner, A. Fromm, T. Kruse & A.P. (2018).

- Determination of the asymptotic behaviour, characterization in terms of a PDE or a BSDE and fixed point argument.
  - P. Graewe, U. Horst & E. Séré (2017).
  - P. Graewe, U. Horst (2017).

Outline

Game of optimal liquidation between N players.

Transaction price for each player $i = 1, \dots, N$

$$S_t^i = S_t^0 - \int_0^t \kappa_s^i \left(\frac{1}{N} \sum_{j=1}^N \xi_s^j \right) ds - \eta_t^i \xi_t^i.$$

Optimization problem of player $i = 1, \dots, N$: minimize

$$J^{N,i}\left(\vec{\xi}\right) = \mathbb{E} \int_0^T \left[\kappa_t^i \left(\frac{1}{N} \sum_{j=1}^N \xi_t^j \right) X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \right] dt$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i \, dt, \ X_0^i = x^i \quad \text{and} \quad X_T^i = 0.$$

$\vec{\xi} = (\xi^1, \cdots, \xi^N)$: vector of strategies of each player.

Game with asymmetric information.

Probabilistic setting:

- $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ be a probability space.
- Carries independent standard Brownian motions $W^0, W^1, ..., W^N$.

Filtrations:

$$\mathbb{F}^i := (\mathcal{F}^i_t, 0 \leq t \leq T), \quad \text{with} \quad \mathcal{F}^i_t := \sigma(W^0_s, W^i_s, 0 \leq s \leq t).$$

Assumptions on the processes $(\kappa^i, \eta^i, \lambda^i)$

- Progressively measurable with respect to the augmented σ-field $\mathbb{F}^i$.
- Conditionally independent and identically distributed, given W^0.

Literature.

Probabilistic approach for MFGs:

- R. Carmona & F. Delarue (2013): stochastic maximum principle and McKean-Vlasov FBSDEs.
- R. Carmona, F. Delarue & D. Lacker (2016): MFGs with common noise.
- R. Carmona, F. Delarue (2018): Probabilistic Theory of Mean Field Games with Applications I-II.

Closest papers:

- R. Carmona & D. Lacker (2015).
- X. Huang, S. Jaimungal & M. Nourian (2015).
- P. Cardaliaguet & C. Lehalle (2017).

Novelty

- Private information and common noise.
- Interaction through the impact of their strategies.
- Terminal constraint.

Plan

Formal problem.

1. Fix a $\mathbb{F}^0$ progressively measurable process μ (in some suitable space).
   - $\mathbb{F}^0 := (\mathcal{F}^0_t, 0 \leq t \leq T)$ with $\mathcal{F}^0_t = \sigma(W^0_s, 0 \leq s \leq t)$.
2. Solve the parameterized constrained optimization problem:

$$\inf_{\xi} \mathbb{E}\left[\int_0^T \left(\kappa_s \mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds\right]$$

s.t.

$$dX_t = -\xi_t \, dt, \; X_0 = x \quad \text{and} \quad X_T = 0.$$

   - W^0 and W are independent.
   - $\mathbb{F} := (\mathcal{F}_t, 0 \leq t \leq T)$ with $\mathcal{F}_t := \sigma(W^0_s, W_s, 0 \leq s \leq t)$.
   - κ, η and λ are $\mathbb{F}$ progressively measurable.
3. Search for the fixed point

$$\mu_t = \mathbb{E}[\xi^*_t | \mathcal{F}^0_t], \;\text{ for a.e. } t \in [0, T],$$

where ξ^* is the optimal strategy of the second step.

Probabilistic approach

Notation: for a filtration $\mathbb{G}$

$$L^p_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \left\{ u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \mathbb{E}\left(\int_0^T |u(s,\omega)|^2 ds\right)^{p/2} < \infty \right\};$$

$$S^p_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) = \left\{ u \in \mathcal{P}_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I});\ \mathbb{E}\left(\sup_{0\le s\le T} |u(s,\omega)|^p\right) < \infty \right\}.$$

A control ξ is admissible if $\xi \in \mathcal{A}_{\mathbb{F}}(t,x)$ with

$$\mathcal{A}_{\mathbb{F}}(t,x) := \left\{ \xi \in L^2_{\mathbb{F}}([t,T]\times\Omega),\ \int_t^T \xi_s\, ds = x \right\}.$$

For a given $\mu \in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R})$, value function

$$V(t,x;\mu) := \inf_{\xi\in\mathcal{A}_{\mathbb{F}}(t,x)} \mathbb{E}\left[\int_t^T \left(\kappa_s \mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds \middle| \mathcal{F}_t \right].$$

Probabilistic approach

Stochastic maximum principle: characterization in terms of the FBSDE

$$\begin{cases} X_s = x - \int_t^s \xi_u \, du & \text{(forward dynamics)}, \\ Y_s = Y_\tau + \int_s^\tau (\kappa_u \mu_u + 2\lambda_u X_u) \, du - \int_s^\tau Z_u \, d\widetilde{W}_u, \\ \qquad \text{(backward dynamics)}, \\ X_T = 0 & \text{(terminal constraint)}. \end{cases}$$

with $t \leq s \leq \tau < T$ and $\widetilde{W} = (W^0, W)$ a Brownian motion.

Remark:

- Y_T cannot be determined a priori. It is implicitly encoded in the FBSDE.
- No a priori sign assumption $\longrightarrow$ penalization method fails.
- The first equation holds on $[0, T]$, the second equation holds on $[0, T)$.

Conditional mean-field type FBSDE.

Standard approach yields the candidate optimal control

$$\xi_s^* = \frac{Y_s}{2\eta_s}.$$

MFG $\longrightarrow$ conditional mean-field type FBSDE

$$\begin{cases} dX_s = -\dfrac{Y_s}{2\eta_s}\, ds, \\ -dY_s = \left(\kappa_s \mathbb{E}\left[\dfrac{Y_s}{2\eta_s} \middle| \mathcal{F}_s^0 \right] + 2\lambda_s X_s \right) ds - Z_s\, d\widetilde{W}_s, \\ X_t = x \\ X_T = 0. \end{cases} \tag{1}$$

Plan

Partial decoupling field.

Ansatz: $Y = AX + B$ where A solves a singular BSDE

$$\begin{cases} -dA_s = \left(2\lambda_s - \frac{A_s^2}{2\eta_s}\right) ds - Z_s^A \, d\widetilde{W}_s, \\ A_T = \infty. \end{cases} \tag{2}$$

and (X, B) satisfies the FBSDE

$$\begin{cases} dX_s = -\frac{1}{2\eta_t}(A_s X_s + B_s)\, ds, \\ -dB_s = \left(\kappa_s \mathbb{E}\left[\frac{1}{2\eta_s}(A_s X_s + B_s) \middle| \mathcal{F}_s^0\right] - \frac{A_s B_s}{2\eta_s}\right) ds - Z_s^B \, d\widetilde{W}_s, \\ X_0 = x \\ B_T = 0. \end{cases} \tag{3}$$

Spaces of weighted stochastic processes.

For $\nu \in \mathbb{R}$,

$$\mathcal{H}_\nu := \{Y : \ (T - .)^{-\nu} Y_. \in S^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R} \cup \{\infty\})\}$$

is endowed with the norm

$$\|Y\|^2_{\mathcal{H}_\nu} := \|Y\|^2_\nu := \mathbb{E}\left[\sup_{0 \leq s \leq T} \left| \frac{Y_s}{(T-s)^\nu} \right|^2 \right].$$

- If $K \in \mathcal{H}_\nu$, with $\nu > 0$, then $K_T = 0$ a.s.

$$\mathcal{M}_\nu := \{Y : \ (T - .)^{-\nu} Y_. \in L^\infty_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})\}$$

is endowed with the norm

$$\|Y\|_{\mathcal{M}_\nu} := \operatorname*{esssup}_{(s,\omega) \in [0,T] \times \Omega} \frac{|Y_s|}{(T-s)^\nu}.$$

Facts:

- If $K_1 \in \mathcal{M}_{-1}$ and $K_2 \in \mathcal{H}_\nu$, then $K_1 K_2 \in \mathcal{H}_{-1+\nu}$.

Setting on the cost coefficients.

Assumption: κ, λ, $\frac{1}{\lambda}$, η and $\frac{1}{\eta}$ belong to $L^\infty_{\mathbb{F}}([0,T] \times \Omega; [0,\infty))$.

Notations:

- $\|\lambda\|$, $\|\kappa\|$, $\|\eta\|$ the bounds of the respective cost coefficients.
- $\lambda_\star$ and $\eta_\star$ the lower bounds of λ and η respectively.
- $$\alpha := \frac{\eta_\star}{\|\eta\|} \in (0,1].$$

Technical condition:

$$16\eta_\star\lambda_\star > \|\kappa\|^2.$$

The singular process A.

From AJK-2014 and GHS-2017

Lemma

In $L^2_{\mathbb{F}}(\Omega; C[0,T-]) \times L^2_{\mathbb{F}}([0,T-];\mathbb{R}^m)$ *there exists a unique solution to* (2)

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t, \\ A_T = \infty. \end{cases}$$

Moreover

$$0 \le \frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s} ds \middle| \mathcal{F}_t \right]} \le A_t \le \frac{1}{(T-t)^2} \mathbb{E}\left[\int_t^T 2\eta_s + 2(T-s)^2 \lambda_s \, ds \middle| \mathcal{F}_t \right].$$

The singular process A.

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Lemma

In $L^2_{\mathbb{F}}(\Omega; C[0,T-]) \times L^2_{\mathbb{F}}([0,T-];\mathbb{R}^m)$ *there exists a unique solution to* (2)

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t, \\ A_T = \infty. \end{cases}$$

Consequences:

- $A \in \mathcal{M}_{-1}$.
- For any $0 \leq r \leq s < T$, with $\alpha = \eta_\star / \|\eta\|$

$$\exp\left(- \int_r^s \frac{A_u}{2\eta_u} du \right) \leq \left(\frac{T-s}{T-r} \right)^\alpha$$

Plan

First result.

Let $0 < \gamma < (1/2) \wedge \alpha$.

Theorem

There exists a unique solution (X, B, Y, Z^B, Z^Y) to the FBSDEs (1) and (3) s.t.

- $X \in \mathcal{H}_\alpha$, $B \in \mathcal{H}_\gamma$;
- $Y \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \cap S^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R})$;
- $(Z^B, Z^Y) \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$.

There exists a constant $C > 0$ depending on η, λ, κ, T and x, s.t.

$$\|X\|_\alpha + \|B\|_\gamma + \mathbb{E}\left[\int_0^T |Y_t|^2 \, dt\right] \leq C.$$

Proof based on continuation method.

Optimal liquidation strategy & equilibrium for the MFG.

Candidates for the optimal portfolio process and the optimal trading strategy:

$$X_t^* = xe^{-\int_0^t \frac{A_r}{2\eta_r} dr} - \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r} dr} ds,$$

$$\xi_t^* = xe^{-\int_0^t \frac{A_r}{2\eta_r} dr} \frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} - \frac{A_t}{2\eta_t} \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r} dr} ds.$$

Theorem

The process ξ^* is an optimal control. Hence $\mu^* = \mathbb{E}[\xi^* | \mathcal{F}^0]$ is the solution to the MFG. Moreover, the value function is given by

$$V(t, x; \mu^*) = \frac{1}{2} A_t x^2 + \frac{1}{2} B_t x + \frac{1}{2} \mathbb{E}\left[\int_t^T \kappa_s X_s^* \xi_s^* \, ds \middle| \mathcal{F}_t \right].$$

Remark:

$$\lim_{t \uparrow T} V(t, x; \mu^*) = \begin{cases} 0, x = 0; \\ \infty, x \neq 0. \end{cases}$$

Outline

Stronger assumption.

The market depth and the risk aversion parameter depend only on the common noise.

- $\eta^i, \lambda^i \in L^\infty_{\mathbb{F}^0}([0,T] \times \Omega; [0,\infty))$.

The processes κ^i satisfy

$$\kappa^i \in L^\infty_{\mathbb{F}^i}([0,T] \times \Omega; [0,\infty)), \quad i = 1, \ldots, N$$

and they admit a common upper bound $\|\kappa\|$

From the previous part.

Benchmark cost functionals

$$J^i(\xi;\mu) := \mathbb{E}\left[\int_0^T \kappa^i_t \mu_t X^i_t + \eta^i_t (\xi^i_t)^2 + \lambda^i_t (X^i_t)^2 \, dt\right].$$

Optimality

$$J^i(\xi;\mu^{*,i}) \geq J^i(\xi^{*,i};\mu^{*,i}),$$

for any $\xi \in L^2_{\mathbb{F}^i}([0,T]\times\Omega;\mathbb{R})$, where

$$\begin{aligned}\xi^{*,i} &= \frac{A^i X^{*,i} + B^{*,i}}{2\eta^i} \in L^2_{\mathbb{F}^i}([0,T]\times\Omega;\mathbb{R}) \\ \mu^{*,i}_t &= \mathbb{E}\left[\xi^{*,i}_t \middle| \mathcal{F}^0_t\right], \; t \in [0,T)\end{aligned}$$

and $(X^{*,i}, B^{*,i}, A^i)$ are the solutions to the system (2) and (3), with κ, η, λ and W replaced by κ^i, η^i, λ^i and W^i, respectively.

MFG equilibrium.

Define

$$\widetilde{\kappa}_t = \mathbb{E}[\kappa_t^i | \mathcal{F}_t^0] = \mathbb{E}[\kappa_t^j | \mathcal{F}_t^0].$$

and

$$\begin{cases} -dA_t = \left(2\lambda_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, dW_t^0, \\ d\widetilde{X}_t = -\dfrac{A_t \widetilde{X}_t + \widetilde{B}_t}{2\eta_t} \, dt \\ -d\widetilde{B}_t = \left(\dfrac{\widetilde{\kappa}_t A_t}{2\eta_t} \widetilde{X}_t + \dfrac{\widetilde{\kappa}_t}{2\eta_t} \widetilde{B}_t - \dfrac{A_t \widetilde{B}_t}{2\eta_t}\right) dt - \zeta_t \, dW_t^0, \\ A_T = \infty, \ \widetilde{X}_0 = x, \ \widetilde{B}_T = 0. \end{cases}$$

Proposition

It holds for each $i = 1, ..., N$ that a.s. a.e.

$$\mu_t^{*,i} = \mu_t^* = \frac{A_t \widetilde{X}_t}{2\eta_t} + \frac{\widetilde{B}_t}{2\eta_t}.$$

ε-Nash equilibrium.

Theorem

Assume that the admissible control space for each player $i = 1, ..., N$ is given by

$$\mathcal{A}^i := \left\{ \xi \in \mathcal{A}_{\mathbb{F}^i}(0, x) : \mathbb{E}\left[\int_0^T |\xi_t|^2 \, dt\right] \leq M \right\}$$

for some fixed positive constant M large enough. Then it holds for each $1 \leq i \leq N$ and each $\xi^i \in \mathcal{A}^i$ that

$$J^{N,i}\left(\vec{\xi^*}\right) \leq J^{N,i}(\xi^{*,-i}, \xi^i) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $(\xi^{*,-i}, \xi^i) = (\xi^{*,1}, \cdots, \xi^{*,i-1}, \xi^i, \xi^{*,i+1}, \cdots, \xi^{*,N})$.

Outline

Unconstrained MFGs.

For a given integer n

1. Fix a process μ;
2. Solve the standard optimization problem: minimize

$$J^n(\xi;\mu) = \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2\right) dt + nX_T^2\right]$$

   such that

$$dX_t = -\xi_t \, dt \quad X_0 = x;$$

3. Solve the fixed point equation :

$$\mu_t^* = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0] \; a.e. \; t \in [0,T],$$

   where ξ^* is the optimal strategy from step 2.

Assumptions.

There exists a constant C such that for any $0 \leq r \leq s < T$

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u} du\right) \leq C\left(\frac{T-s}{T-r}\right)$$

(With the former notation, $\alpha = 1$).

Lemma

The previous assumption holds under each of the following conditions:

- η *is deterministic ;*
- $1/\eta$ *is a positive martingale ;*
- $1/\eta$ *has uncorrelated multiplicative increments, namely for any* $0 \leq s \leq t$

$$\mathbb{E}\left[\frac{\eta_s}{\eta_t}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{\eta_s}{\eta_t}\right].$$

Related conditional mean field FBSDE.

$$\begin{cases} dX_t^n = \left(-\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\right) dt, \\ -dB_t^n = \left(-\dfrac{A_t^n B_t^n}{2\eta_t} + \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t} \middle| \mathcal{F}_t^0\right]\right) dt - Z_t^{B^n}\, d\widetilde{W}_t, \\ dY_t^n = \left(-2\lambda_t X_t^n - \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t} \middle| \mathcal{F}_t^0\right]\right) dt + Z_t^{Y^n}\, d\widetilde{W}_t, \\ X_0^n = x, \\ B_T^n = 0, \\ Y_T^n = 2nX_T^n, \end{cases} \tag{4}$$

where

$$-dA_t^n = \left\{2\lambda_t - \frac{(A_t^n)^2}{2\eta_t}\right\} dt - Z_t^{A^n}\, d\widetilde{W}_t, \quad A_T^n = 2n. \tag{5}$$

Theorem

There exists a unique solution $(X^n, B^n, Y^n, Z^{B^n}, Z^{Y^n})$ in $\mathcal{H}_\alpha^n \times \mathcal{H}_\gamma^n \times S_{\mathbb{F}}^2([0,T]\times\Omega;\mathbb{R}) \times L_{\mathbb{F}}^2([0,T]\times\Omega;\mathbb{R}^m) \times L_{\mathbb{F}}^2([0,T]\times\Omega;\mathbb{R}^m)$.

Approximation

Lemma

There exists a constant $C > 0$ such that

$$\|X^n\|_{n,\alpha} + \|B^n\|_{n,\gamma} + \mathbb{E}\left[\int_0^T |Y_t^n|^2 \, dt\right] \leq C,$$

for any n.

Lemma

If $\alpha = 1$, then

$$\lim_{n\to+\infty} \left\{ \mathbb{E}\left[\int_0^T |X_t^n - X_t^*|^2 \, dt\right] + \mathbb{E}\left[\int_0^T |B_t^n - B_t^*|^2 \, dt\right] + \mathbb{E}\left[\int_0^T |Y_t^n - Y_t^*|^2 \, dt\right] \right\} = 0.$$

Approximation

Lemma

If $\alpha = 1$, then

$$\lim_{n \to +\infty} \left\{ \mathbb{E}\left[\int_0^T |X_t^n - X_t^*|^2 \, dt \right] + \mathbb{E}\left[\int_0^T |B_t^n - B_t^*|^2 \, dt \right] + \mathbb{E}\left[\int_0^T |Y_t^n - Y_t^*|^2 \, dt \right] \right\} = 0.$$

Theorem

The value function $V^n(x)$ converges to $V(x)$.

Thank you for your attention !

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Closest papers

- P. Cardaliaguet and C. Delalle. Mean Field Game of Controls and An Application To Trade Crowding. Mathematics and Financial Economics, 2017.
- R. Carmona and D. Lacker. A Probabilistic Weak Formulation of Mean Field Games and Applications. Annals of Applied Probability, 2015.
- X. Huang, S. Jaimungal and M. Nourian. Mean-Field Game Strategies for Optimal Execution. SSRN, 2015.

Benchmark case.

- All the randomness is generated by the Brownian motion W^0 that drives the benchmark price process.
- All players share the same information.

Assumption: κ, λ, η and $1/\eta$ belong to $L^\infty_{\mathbb{F}^0}([0,T]\times\Omega;[0,\infty))$.

- Consistency condition : $\mu = \xi^*$.
- Conditional mean-field FBSDE:

$$\begin{cases} dX_t = -\dfrac{Y_t}{2\eta_t}\,dt, \\ -dY_t = \left(\dfrac{\kappa_t Y_t}{2\eta_t} + 2\lambda_t X_t\right)\,dt - Z_t\,dW^0_t, \\ X_0 = x, \\ X_T = 0. \end{cases}$$

Benchmark case.

Linear ansatz $Y = AX$:

$$-dA_t = \left(2\lambda_t + \frac{\kappa_t A_t}{2\eta_t} - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A dW_t^0, \quad A_T = \infty.$$

Lemma

This equation has a unique solution. Moreover the processes A, $X_t^* = xe^{-\int_0^t \frac{A_r}{2\eta_r} dr}$, $Y = AX^*$ *and* $\xi^* = \mu = \frac{Y}{2\eta}$ *are all non negative and*

$$A \in \mathcal{M}_{-1}, \; X^* \in \mathcal{M}_{\alpha}, \; Y \in \mathcal{M}_{\alpha-1}, \; \xi^* \in \mathcal{M}_{\alpha-1}.$$

Theorem

$\xi^*(= \mu^*)$ is an admissible optimal control as well as the equilibrium to MFG. Moreover the value function is given by:

$$V(t, x; \mu^*) = \frac{1}{2} A_t x^2 + \frac{1}{2}\mathbb{E}\left[\int_t^T \kappa_s \mu_s^* X_s^* \, ds \,\middle|\, \mathcal{F}_t^0\right].$$

Deterministic benchmark example.

$T = 1$, $x = 1$, $\lambda = 5$ and $\eta = 5$.

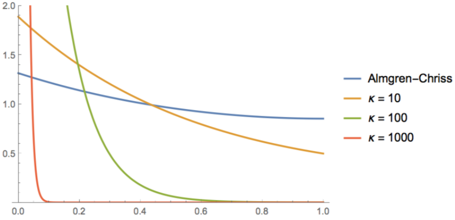

Figure: Trading rate ξ^*

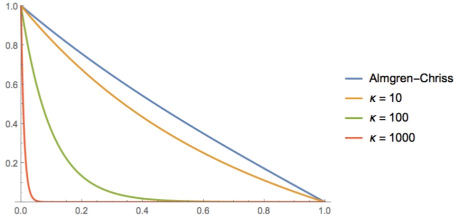

Figure: Position X^*

Almgren-Chriss model: $\kappa = 0$ = no interaction.