BCQT – Bicategories – Lecture 2 – Enriched categories

3) ENRICHED CATEGORIES

DEFN Let $\mathcal{V}$ be a monoidal caty. A $\mathcal{V}$-enriched category $\mathcal{C}$ (or $\mathcal{V}$-category) involves:

- a collection of objects $ob(\mathcal{C})$
- for $x, y \in ob(\mathcal{C})$, a hom-object $\mathcal{C}(x,y) \in \mathcal{V}$
- for $x,y,z \in ob(\mathcal{C})$, a composition map
$$m_{xyz} : \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z) \quad \text{in } \mathcal{V}$$
- for $x \in ob(\mathcal{C})$, an identities map
$$e_x : I \longrightarrow \mathcal{C}(x,x)$$

satisfying associativity and unitality axioms.

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Just as a caty is a "many object monoid", so a $\mathcal{V}$-caty is a "many-object monoid in $\mathcal{V}$".

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Examples

- A $\underline{Set}$-caty is just a (locally small) ordinary category;
- A $\underline{kVect}$-caty is a k-linear caty;
- A $[\Delta^{op}, Set]$-caty is a simplicially enriched category;
- A $\underline{Cat}$-caty is a 2-category;
- A $\underline{Ch(k\text{-}Vect)}$-caty is a dg-category;

In all of these examples, a $\mathcal{V}$-caty is just an ordinary caty which has been "enhanced" somehow. The following makes this precise.

DEFN Let $\mathcal{C}$ be a $\mathcal{V}$-caty. The underlying ordinary category of $\mathcal{C}$ is the caty $\mathcal{C}_0$ with the same objects, and

$$\frac{\text{arrows } x \longrightarrow y \text{ in } \mathcal{C}_0}{\text{maps } I \longrightarrow \mathcal{C}(x,y) \text{ in } \mathcal{V}}$$

In this situation, say that $\mathcal{C}$ is a $\mathcal{V}$-enrichment of $\mathcal{C}_0$.

Exercise: what are $\mathcal{V}^{\mathbb{N}}$-, $\mathcal{V}^{I \times I}$-, $\mathcal{V}[\varepsilon]$-catys concretely?

Let's look at some novel examples:

Ex A caty enriched in $(\mathrm{Set}^{\rightarrow}, \times, 1)$ has an obj. set $\mathrm{ob}(\mathcal{C})$; for $x,y \in \mathcal{C}$ a pair of homsets $\mathcal{C}_0(x,y) \xrightarrow{J_{xy}} \mathcal{C}_1(x,y)$. With a little thought: comp + identities provide caty structure on $\mathcal{C}_0$ and $\mathcal{C}_1$, so that $J : \mathcal{C}_0 \to \mathcal{C}_1$ is an identity on objs functor. So a $(\mathrm{Set}^{\rightarrow})$-caty is a pair of catys $\mathcal{C}_0, \mathcal{C}_1$ with an i.o.o. functor between them.

Ex Consider Subset whose objs are pairs $(X \in \mathrm{Set}, U \subseteq X)$, and maps are f$^{\text{n}}$s preserving the subset. This has a monoidal structure:

$$(U \subseteq X) \otimes (V \subseteq Y) = \left(U \times Y \cup X \times V\right) \subseteq X \times Y$$

A (Subset)-caty is a caty $\mathcal{C}_0$ together with a subset $I \subseteq \mathrm{mor}(\mathcal{C}_0)$ constituting a two-sided ideal; ie, $f \in I \Rightarrow hf, fg \in I$ for all suitable h, g.

Ex Let $\mathcal{C}$ be a category, T a monad on $\mathcal{C}$. We can give an enrichment $\underline{\mathcal{C}}$ of $\mathcal{C}$ in $[\Delta_+^{op}, \mathrm{Set}]_{\mathrm{conv}}$ given by

$$\underline{\mathcal{C}}(x,y) : \Delta_+^{op} \longrightarrow \mathrm{Set}$$
$$\underline{n} \longmapsto \mathcal{C}(T^n x, y)$$

Note: presheaf structure uses the monad structure of T.

Composition is

$$\underline{\mathcal{C}}(y,z)(\underline{m}) \times \mathcal{C}(x,y)(\underline{n}) \longrightarrow \underline{\mathcal{C}}(x,z)(\underline{m} \oplus \underline{n})$$
$$f : T^m y \to z, \quad g : T^n x \to y \longmapsto T^{m+n} x \xrightarrow{T^m g} T^m y \xrightarrow{f} z.$$

DEFN A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-catgys involves:

- a mapping $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$
- for all $x, y \in \mathrm{ob}(\mathcal{C})$, a map $F_{xy} : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ in $\mathcal{V}$
- plus functoriality axioms

For example:

- A $\underline{\mathrm{Set}}$-functor is a functor;
- A $\underline{\mathrm{Cat}}$-functor is a 2-functor (= strict homomorphism)

- A $k\underline{\mathrm{Vect}}$-functor is a k-linear functor.

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What about our more interesting examples?

- A $\underline{\mathrm{Set}}^{\to}$-functor $\mathcal{C} \to \mathbb{D}$ is a pair of functors in a commuting square

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, "i\cdot o\cdot o\cdot"'] & \mathbb{D}_0 \arrow[d] \\ \mathcal{C}_1 \arrow[r, "F_1"'] & \mathbb{D}_1 \end{tikzcd}$$

- A $\underline{\mathrm{Subset}}$-functor $\mathcal{C} \to \mathbb{D}$ is a functor $F : \mathcal{C} \to \mathbb{D}$ st $F(I_{\mathcal{C}}) \subseteq I_{\mathbb{D}}$.
- If T, S are monads on $\mathcal{C}, \mathbb{D}$, and $\underline{\mathcal{C}}, \underline{\mathbb{D}}$ are the associated $[\Delta_+^{op}, \mathrm{Set}]$-catys, then a $[\Delta_+^{op}, \mathrm{Set}]$-functor $\underline{\mathcal{C}} \to \underline{\mathbb{D}}$ is an ordinary functor $F : \mathcal{C} \to \mathbb{D}$ t/w a nat xfm $\delta : SF \Rightarrow FT$ compatible with monad structures. (A <u>monad morphism</u> $(\mathcal{C}, T) \to (\mathbb{D}, S)$).

  In particular, a $[\Delta_+^{op}, \mathrm{Set}]$-functor $1 \to \underline{\mathbb{D}}$ is precisely an S-algebra in $\mathbb{D}$.

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There's also a notion of <u>$\mathcal{V}$-natural transformation</u> $\alpha : F \Rightarrow G : \mathcal{C} \to \mathbb{D}$. between $\mathcal{V}$-functors. Such an α involves a family of <u>components</u> $\alpha_x : Fx \to Gx$ in $\mathbb{D}_0$, satisfying a <u>$\mathcal{V}$-naturality</u> condition (see notes).

In this way, we get a 2-caty of $\mathcal{V}$-catys, $\mathcal{V}$-functors + $\mathcal{V}$-nat transformations.

4) THE FUNDAMENTAL THEOREM OF ENRICHED CATEGORIES

DEFN Let $\mathcal{V}$ be a monoidal caty. A $\mathcal{V}$-action on a caty $\mathcal{C}$ is a functor $\bullet : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ tw nat isoms.

$$\lambda : I \bullet C \xrightarrow{\cong} C \quad \text{and} \quad \alpha : (V \otimes W) \bullet C \xrightarrow{\cong} V \bullet (W \bullet C)$$

plus axioms. Equiv: it's a strong monoidal functor $(\mathcal{V}, \otimes) \longrightarrow ([\mathcal{C}, \mathcal{C}], \circ)$.

Ex
- $\mathcal{V}$ acts on itself via $\otimes$.
- If $\mathcal{V} \xrightarrow{F} \mathcal{W}$ is strong monoidal functor, then any $\mathcal{W}$-action on $\mathcal{C}$ yields a $\mathcal{V}$-action by precomposition.
- If $\mathcal{C}$ has coproducts, have an action
$$\begin{aligned} \mathrm{Set} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ I, C &\longmapsto I \bullet C = \sum_{i \in I} C \end{aligned}$$
- Have an action $k\text{-Vect} \times A\text{-mod} \longrightarrow A\text{-mod}$ ($\leftarrow$ k-algebra)
$$(V, M) \longmapsto V \otimes_k M$$
- Have an action $k\text{-Coalg} \times k\text{-Alg}^{op} \longrightarrow k\text{-Alg}^{op}$
$$C, A \longmapsto \mathrm{Hom}(C, A)$$
- A monad on $\mathcal{C}$ is same as a strict monoidal functor $\Delta_+ \longrightarrow [\mathcal{C}, \mathcal{C}]$. So we get a strictly assoc. and unital action $\Delta_+ \times \mathcal{C} \to \mathcal{C}$.
$$(\underline{n}, C) \mapsto T^n C.$$

DEFN A monoidal action $\bullet$ is right closed if each functor $(-)\bullet C : \mathcal{V} \to \mathcal{C}$ has a right adjoint $\langle C, - \rangle : \mathcal{C} \to \mathcal{V}$, ie:

$$\frac{V \bullet C \longrightarrow D \quad \text{in } \mathcal{C}}{V \longrightarrow \langle C, D \rangle \quad \text{in } \mathcal{V}}$$

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For example:

- Self-action $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is right-closed just when $\mathcal{V}$ is a right-closed monoidal cat^y: true in almost all of our examples!
- $k\text{-Vect} \times A\text{-Mod} \longrightarrow A\text{-Mod}$ is right closed, where $\langle M, N \rangle$ = k-vect sp. of A-Mod homomorphisms. for $M, N \in A\text{-Mod}$
- $k\text{-Coalg} \times k\text{-Alg}^{op} \longrightarrow k\text{-Alg}^{op}$ is right-closed, where for $A, B \in k\text{-Alg}$, $\langle A, B \rangle$ is Sweedler's measuring coalgebra from A to B; satisfies

$$\frac{B \longrightarrow \mathrm{Hom}(C, A) \quad \text{in } k\text{-Alg}}{C \longrightarrow \langle B, A \rangle \quad \text{in } k\text{-Coalg}.}$$

- $\mathbf{Set} \times \mathcal{C} \longrightarrow \mathcal{C}$ is always right-closed, with $(I, C) \longmapsto I \bullet C$ $\qquad \langle C, D \rangle = \mathcal{C}(C, D).$

THEOREM (Fundamental thm of enriched caty theory) Any right-closed action $\bullet : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ gives rise to an enrichment of $\mathcal{C}$ in $\mathcal{V}$. Moreover, the $\mathcal{V}$-catys arising in this way are precisely those which have copowers.

What's a copower?

DEFN Let $\mathcal{C}$ be a $\mathcal{V}$-caty, $C \in \mathcal{C}$, $V \in \mathcal{V}$. A copower of C by V is an object $V \bullet C$ in $\mathcal{C}$ t/w a map $V \to \mathcal{C}(C, V \bullet C)$ in $\mathcal{V}$, composition with which induces isomorphisms

$$\begin{array}{ll} \text{in } \mathcal{V} & A \otimes V \longrightarrow \mathcal{C}(C, D) \\ \hline \text{in } \mathcal{V} & A \longrightarrow \mathcal{C}(V \bullet C, D) \end{array} \qquad \text{so in ptic} \qquad \begin{array}{ll} V \longrightarrow \mathcal{C}(C, D) & \text{in } \mathcal{V} \\ \hline V \bullet C \longrightarrow D & \text{in } \mathcal{C}_0 \end{array} \quad \circledast$$

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Proof of thm

Given a $\mathcal{V}$-action $\bullet : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ which is right-closed, we define a $\mathcal{V}$-enrichment of $\mathcal{C}$, say $\underline{\mathcal{C}}$, where:

$$\underline{\mathcal{C}}(C, D) = \langle C, D \rangle \quad \leftarrow \text{from right-closure of } \bullet$$

Identities:
$$\frac{I \bullet C \xrightarrow{\cong} C}{I \xrightarrow[e_C]{} \langle C, C \rangle}$$

Composition
$$\frac{(\langle D, E\rangle \otimes \langle C, D \rangle) \bullet C \cong \langle D, E \rangle \bullet (\langle C, D \rangle \bullet C) \xrightarrow{1 \bullet ev} \langle D, E \rangle \bullet D \xrightarrow{ev} E}{\langle D, E \rangle \otimes \langle C, D \rangle \longrightarrow \langle C, E \rangle}$$

Easy to check the axioms; now the unit maps $V \to \langle C, V \cdot C\rangle$ of the adjs $(-) \cdot C \dashv \langle C, -\rangle$ exhibit $V \cdot C$ as a copower of C by V in $\underline{\mathcal{C}}$.

Conversely, if $\underline{\mathcal{C}}$ is a $\mathcal{V}$-caty with copowers, then taking copowers gives a $\mathcal{V}$-action on $(\underline{\mathcal{C}})_0 = \mathcal{C}$:

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\begin{aligned}
\mathcal{V} \times \mathcal{C} &\longrightarrow \mathcal{C} \\
V, C &\longmapsto V \cdot C \; ,
\end{aligned}
$$

which is right closed by ⊛, with $\langle C, D\rangle = \underline{\mathcal{C}}(C, D)$. Π.

For example:

- if $\mathcal{V}$ is right-closed monoidal caty, then it has an enrichment over itself: for example:
  - $\underline{Cat}$ is a Cat-category (ie a 2-caty)
  - $[\Delta^{op}, Set]$ is a simplicially enriched caty
  - k-Vect is a k-linear category
- Via $Set \times \mathcal{C} \to \mathcal{C}$ ($\mathcal{C}$ caty with coprods), get an enrichment of $\mathcal{C}$ over $\underline{Set}$... which is just $\mathcal{C}$!
- Via $k\text{-Vect} \times A\text{-Mod} \to A\text{-Mod}$, we see that A-Mod is a k-linear caty.

- Via $k\text{-Coalg} \times k\text{-Alg}^{op} \longrightarrow k\text{-Alg}^{op}$, get an enrichment of $k\text{-Alg}^{op}$ in $k\text{-Coalg}$.