BCQT Summer School — Bicategories

Lecture 1 - Monoidal categories + bicategories

1) MONOIDAL CATEGORIES

A *monoid* is a set M t/w a constant $1 \in M$ and operation $n, m \mapsto n \cdot m$ st:

$$(m \cdot n) \cdot k = m \cdot (n \cdot k) \qquad 1 \cdot m = m = m \cdot 1 \qquad \circledast$$

These axioms are justified by the fact that free monoid on a set X is the monoid X^* of lists of elements of X under concatenation.

To "categorify" this def^{n}, replace the equalities $\circledast$ by *coherent* isomorphisms.

DEFN A monoidal catg is a catg $\mathcal{V}$ t/w a *unit* object $I \in \mathcal{V}$; and a *tensor product* functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and nat. isos. α, λ, ρ with components

$$\alpha_{xyz} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \qquad \lambda_x : I \otimes x \xrightarrow{\sim} x \qquad \rho_x : x \xrightarrow{\sim} x \otimes I$$

satisfying two axioms:

(1)
$$\begin{tikzcd}
 & \cdot \arrow[dr] & & \\
((xy)z)w \arrow[ur] \arrow[d] & & x(y(zw)) \\
\cdot \arrow[r] & \cdot \arrow[ur] &
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
(xI)y \arrow[r] & x(Iy) \arrow[d] \\
xy \arrow[u] \arrow[r, equal] & xy
\end{tikzcd}$$

These axioms are justified by fact that free monoidal caty on an X-indexed family of objects is <u>equivalent</u> to the discrete caty X^*.

<u>Examples</u>

- $(\mathrm{Set}, \times, 1)$
- $(\mathcal{C}, \times, 1)$ where $\mathcal{C}$ any caty w/ products:
  eg: Cat; $[\Delta^{op}, \mathrm{Set}]$; Top; $\mathrm{Sh}(X)$; ...
- $(k\text{-Vect}, \otimes, k)$; similarly $(\mathrm{Ab}, \otimes, \mathbb{Z})$, $(\vee\text{-SLat}, \otimes, 2)$
- $(\mathrm{Rep}(G), \otimes, k)$; more generally, modules over any $\left\{\begin{matrix}\text{Hopf algebra} \\ \text{bialgebra}\end{matrix}\right\}$
- $([\mathcal{C}, \mathcal{C}], \circ, \mathrm{id})$ for any caty $\mathcal{C}$
- Restricted functor catys: eg if $\mathcal{C}$ is cocomplete, $(\mathrm{Cocts}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id})$.
  eg: $(\mathrm{Cocts}(\mathrm{Set}, \mathrm{Set}), \circ, \mathrm{id}) \simeq (\mathrm{Set}, \times, \mathrm{id})$
  $(\mathrm{Cocts}(k\mathrm{Vect}, k\mathrm{Vect}), \circ, \mathrm{id}) \simeq (k\mathrm{Vect}, \otimes, k)$
- Small examples:
  - $\Delta_+ =$ (finite ordinals $\underline{n} = \{0, \ldots, n-1\}$), $\oplus =$ ordinal sum.
  - Any monoid M becomes a <u>discrete</u> monoidal caty $(M, \cdot, 1)$
  - Any <u>comm.</u> monoid M becomes a one-object monoidal caty with $\circ = \otimes = \cdot$.

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We can also build new monoidal catys from existing ones.

Obvious: $\mathcal{V} \times \mathcal{W}$ monoidal if $\mathcal{V}, \mathcal{W}$ are.

<u>Less obvious:</u> suppose $\mathcal{V}$ is monoidal w/ coproducts and $\otimes$ preserves coprods in each variable (ie $A \otimes (-)$, $(-) \otimes A : \mathcal{V} \longrightarrow \mathcal{V}$ pres. coprods).

- $\mathcal{V}^{\mathbb{N}}$ is monoidal via $(A \otimes B)_n = \sum_{n=m+k} A_m \otimes B_k$.

- $\mathcal{V}^{I \times I}$ is monoidal for any set I, via $(A \otimes B)_{ik} = \sum_{j \in I} A_{jk} \otimes B_{ij}$

- $\mathcal{V} \times \mathcal{V}$ monoidal via $(A, C) \otimes (B, D) = (A \otimes B,\ A \otimes D + C \otimes B)$
  Call this $\mathcal{V}[\varepsilon]$ by analogy with ring of dual numbers.

- $\mathcal{V}$ itself monoidal via $A \otimes' B = A + B + A \otimes B$ $\qquad I' = 0$
  and $A \otimes'' B = A + B + AB + BA + ABA + BAB + \cdots$ $\qquad I' = 0$.

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A very important way of building mon. strs. is via convolution.

DEFN Let $\mathcal{V}, \mathcal{W}$ be monoidal catgs, $F, G, H : \mathcal{V} \longrightarrow \mathcal{W}$ arbitrary functors.

- A bilinear map $F, G \longrightarrow H$ is a natural family of maps

$$FX \otimes_{\mathcal{W}} GY \longrightarrow H(X \otimes_{\mathcal{V}} Y) \qquad \forall X, Y \in \mathcal{V}$$

- A 0-linear map $() \longrightarrow H$ is a map $I_{\mathcal{W}} \longrightarrow H(I_{\mathcal{V}})$

The Day convolution monoidal str $\otimes_{\mathrm{conv}}$ on $[\mathcal{V}, \mathcal{W}]$, when it exists, is characterized by the fact that

$$\frac{F, G \longrightarrow H \quad \text{bilinear}}{F \otimes_{\mathrm{conv}} G \longrightarrow H \quad \text{nat xfm}} \qquad\qquad \frac{() \longrightarrow H}{I_{\mathrm{conv}} \longrightarrow H}$$

The existence of $\otimes_{\mathrm{conv}}$ is guaranteed if $\mathcal{V}$ small, $\mathcal{W}$ cocomp, and $\otimes$ in $\mathcal{W}$ pres. colims in each variable. The explicit formulae are then:

$$(F \otimes_{conv} G)(V) = \int^{W,X} \mathcal{V}(W \otimes X, V) \cdot FW \otimes GX$$

$$I(V) = \mathcal{V}(I, V) \cdot I$$

For example: • $\mathcal{V}^{\mathbb{N}}$ above is the conv. monoid structure from $(\mathbb{N}, \cdot, 1)$ to $(\mathcal{V}, \otimes, I)$.

• Convolution str. on $[\Delta_+^{op}, Set]$ gives join of augmented simplicial sets.

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<u>DEFN</u> A <u>monoid</u> in a monoidal catg $\mathcal{V}$ is $M \in \mathcal{V}$ t/w $m: M \otimes M \to M$, $e: I \to M$ satisfying unit and assoc. axioms.

<u>Examples</u>: • In Set, Top, $[\Delta^{op}, Set]$: monoid, topological monoid, simplicial monoid.

- In Cat: <u>strict</u> monoidal catg ($\alpha = \lambda = \rho = id$)
- In k-Vect: k-algebra; in <u>Ab</u>: ring
- In $[\mathcal{C}, \mathcal{C}]$ (or $Cocts(\mathcal{C}, \mathcal{C})$): a monad (or cocts monad) on $\mathcal{C}$.
- In Δ_+: $\underline{1} = \{0\}$ is a monoid, in fact, Δ_+ is the free monoidal catg on the monoid $\underline{1}$.
- In $\mathcal{V}^{\mathbb{N}}$: a graded monoid in $\mathcal{V}$: $(M_i)_{i \in \mathbb{N}}$ t/w $M_i \otimes M_j \to M_{i+j}$ $\quad I \to M_0$ + axioms.

- In $\mathcal{V}^{I \times I}$: involves $\mathcal{V}$-objects $(\mathcal{C}(i,j))_{i,j \in I}$ t/w

$$\mathcal{C}(j,k) \otimes \mathcal{C}(i,j) \to \mathcal{C}(i,k) \qquad I \to \mathcal{C}(i,i).$$

+ axioms: next time, we'll see these are <u>$\mathcal{V}$-categories</u> with ob set I.

- In $\mathcal{V}[\varepsilon]$: (A, B) t/w $(A \otimes A, A \otimes B + B \otimes A) \longrightarrow (A, B)$
$(I, 0) \longrightarrow (A, A)$
... monoid A t/w an A-A-bimodule B.

- In $(\mathcal{V}, \otimes', 0)$: $\otimes'$-monoid is a $\otimes$-semigroup (monoid without unit).

What about monoids in $[\mathcal{V}, \mathcal{W}]_{conv}$? Well, its $F: \mathcal{V} \to \mathcal{W}$ t/w

$$F, F \xrightarrow{m} F \qquad () \xrightarrow{e} F$$

ie, maps $\quad FX \otimes FY \xrightarrow{m_{xy}} F(X \otimes Y) \qquad I \xrightarrow{e} FI \quad$ ⊛.
plus axioms. More generally:

<u>DEFN</u> A (lax) <u>monoidal functor</u> $F: \mathcal{V} \to \mathcal{W}$ between mon catgs is a functor $F: \mathcal{V} \to \mathcal{W}$ t/w data ⊛ plus axioms.

A monoidal functor is called <u>strong</u> if each ⊛ is invertible.

Also have oplax mon. functors which involve maps $F(X \otimes Y) \to FX \otimes FY$ and $FI \to I$ plus axioms.

There's a caty MonCat of monoidal caty + monoidal functors, and we can justify the importance of convolution by noting that

$$\frac{\mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{W} \quad \text{monoidal functors}}{\mathcal{U} \longrightarrow [\mathcal{V}, \mathcal{W}]_{\mathrm{conv}} \quad \text{monoidal functors}}$$

2) BICATEGORIES

Just as monoids $\overset{\text{many}}{\underset{\text{objects}}{\rightsquigarrow}}$ categories; so monoidal catys $\overset{\text{many}}{\underset{\text{objects}}{\rightsquigarrow}}$ bicategories

objects: $f : x \to y$ 1-cells

morphisms: $x \underset{g}{\overset{f}{\Downarrow\alpha}} y$ 2-cells

DEFN A bicategory $\mathcal{B}$ comprises:

- a set of objects $ob(\mathcal{B})$;
- for all $x, y \in ob(\mathcal{B})$, a hom-category $\mathcal{B}(x,y)$;
- for all $x,y,z \in ob(\mathcal{B})$, a composition functor
$$\otimes_y : \mathcal{B}(y,z) \times \mathcal{B}(x,y) \longrightarrow \mathcal{B}(x,z)$$
- for all $x \in ob(\mathcal{B})$, an identity 1-cell $I_x \in \mathcal{B}(x,x)$.

plus: nat isos:

$$\begin{tikzcd}
 & x \arrow[r, "N"] \arrow[d, phantom, "\Downarrow\alpha"] & y \arrow[dr, "M"] & \\
(w \arrow[ur, "P"] \arrow[r, "P"'] & (x \arrow[r, "N"'] & y \arrow[r, "M"'] & z)
\end{tikzcd}$$

$$\alpha : (M \otimes_y N) \otimes_x P \longrightarrow M \otimes_y (N \otimes_x P) \quad \text{in} \quad \mathcal{B}(w,z)$$

$$\lambda_M : I_y \otimes_y M \longrightarrow M \qquad \rho_M : M \longrightarrow M \otimes_x I_x$$

satisfying 2 axioms. If $\alpha = \lambda = \rho = \mathrm{id}$, call $\mathcal{B}$ a **2-category**.

Examples

- **Cat** is a 2-category, where 2-cells are nat. transfs.
- **MonCat** is a 2-category, where 2-cells are **monoidal** nat. transfs. Likewise, have $\mathbf{MonCat}_{\text{strong}}$, $\mathbf{MonCat}_{\text{oplax}}$.
- There's a bicaty $\mathrm{Span}(\mathbf{Set})$, where objects are sets, and 1-cells $X \rightsquigarrow Y$ are spans

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}.$$

Composition is by **pullback**:

$$\begin{tikzcd} & & N\otimes_Y M \arrow[dl] \arrow[dr] \arrow[dd] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & M \arrow[dl] \arrow[dr] & & N \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}.$$

This is **not** a 2-caty; only a bicategory.

More generally: $\mathrm{Span}(\mathcal{E})$ for any $\mathcal{E}$ with pullbacks.

- If $\mathcal{V}$ a mon caty w/ coprods preserved by each $A\otimes(-)$, $(-)\otimes A$, get a bicaty $\mathrm{Mat}(\mathcal{V})$ with:
  - obs: sets
  - 1-cells $I \to J$: objects of $\mathcal{V}^{I\times J}$
  - composition of $I \overset{M}{\rightsquigarrow} J \overset{N}{\rightsquigarrow} K$ is

$$(N \otimes_J M)_{ik} = \sum_{j\in J} N_{jk} \otimes M_{ij}.$$

Note: $\mathrm{Mat}(\mathcal{V})(I,I)$ is the $\mathcal{V}^{I\times I}$ we saw before.

In general, if $\mathcal{B}$ any bicaty, then $(\mathcal{B}(x,x), \otimes_x, I_x)$ is monoidal category.

<u>Note</u>: $\mathrm{Span}(\mathrm{Set}) \sim \mathrm{Mat}(\mathrm{Set})$.

- There's a bicaty <u>Bim</u> w/ rings as objects; 1-cells $M: R \rightsquigarrow S$ is a left R- right S- bimodule; composition is tensor product of bimodules.

- There's a 2-caty <u>Rel</u> with:
  - objects sets
  - 1-cell $R: X \to Y$ is a relation $R \subseteq X \times Y$
  - 2-cell $R \Rightarrow S : X \to Y$ is the assertion that $R \subseteq S$.
  - composition is relational composition.

- If $\mathcal{C}$ is any caty, can see it as a 2-caty with only identity 2-cells;

- If $\mathcal{V}$ is a monoidal caty, can see it as a one-object bicategory

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We can now redevelop the preceding ideas for mon catys for bicategories. For example:

- a <u>monad</u> in a bicaty $\mathcal{B}$ on an object $x \in \mathcal{B}$ is a monoid in $(\mathcal{B}(x,x), \otimes_x, I_x)$.
- a <u>morphism</u> of bicatys $F: \mathcal{B} \to \mathcal{C}$ involves:
  - a map on objects: $\mathrm{ob}(\mathcal{B}) \to \mathrm{ob}(\mathcal{C})$;

- functors: $\mathcal{B}(x,y) \xrightarrow{F_{xy}} \mathcal{C}(Fx, Fy)$
- 2-cells: $F_{yz}(M) \otimes_{Fy} F_{xy}(N) \xrightarrow{m} F_{xz}(M \otimes_y N)$

$$I_{Fx} \xrightarrow{e} F(I_x)$$

... yielding a caty **Bicat** of bicatys and morphisms.
(call F a *homomorphism* if each m, e invertible).

- there's a notion of *convolution* bicaty $[\mathcal{B}, \mathcal{C}]_{conv}$ such that

$$\frac{\text{morphisms} \quad \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}}{\text{morphisms} \quad \mathcal{A} \longrightarrow [\mathcal{B}, \mathcal{C}]_{conv}}$$