

BCQT Summer School - Bicategories

Lecture 1 - Monoidal categories + bicategories

1) MONOIDAL CATEGORIES

A monoid is a set M t/w a constant $1 \in M$ and operation $n, m \mapsto n \cdot m$ st:

$$(m \cdot n) \cdot k = m \cdot (n \cdot k) \quad 1 \cdot m = m = m \cdot 1 \quad \textcircled{*}$$

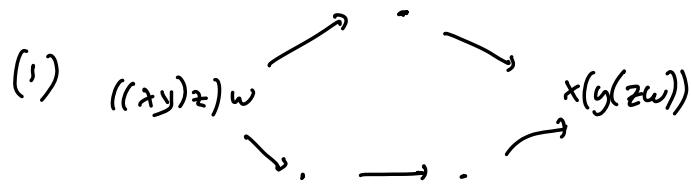
These axioms are justified by the fact that free monoid on a set X is the monoid X^* of list of elements of X under concatenation.

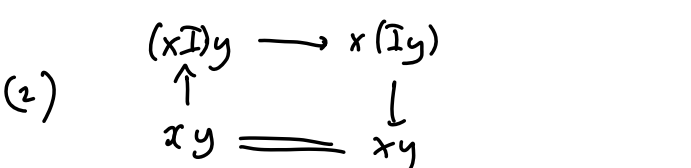
To "categorify" this defⁿ, replace the equalities $\textcircled{*}$ by coherent isomorphisms.

DEFN A monoidal caty is a caty \mathcal{V} t/w a unit object $I \in \mathcal{V}$; and a tensor product functor $\otimes: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$; and nat. isos. α, λ, ρ with components

$$\alpha_{x,y,z}: (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \quad \lambda_x: I \otimes x \xrightarrow{\sim} x \quad \rho_x: x \otimes I \xrightarrow{\sim} x$$

satisfying two axioms:

(1) 

(2) 

These axioms are justified by fact that free monoidal caty on an X -indexed family of objects is equivalent to the discrete caty X^* .

Examples

- $(\text{Set}, \times, 1)$
- $(\mathcal{C}, \times, 1)$ where \mathcal{C} any caty w/ products:
eg: Cat ; $[\Delta^{\text{op}}, \text{Set}]$; Top ; $\text{Sh}(X)$; ...
- $(k\text{-Vect}, \otimes, k)$; similarly $(\text{Ab}, \otimes, \mathbb{Z})$, $(\text{v-stat}, \otimes, \mathbb{Z})$
- $(\text{Rep}(G), \otimes, k)$; more generally, modules over any $\left\{ \begin{array}{l} \text{Hopf algebra} \\ \text{bialgebra} \end{array} \right\}$
- $([\mathcal{C}, \mathcal{C}], \circ, \text{id})$ for any caty \mathcal{C}
- Restricted fincst catys: eg if \mathcal{C} is cocomplete, $(\text{Cocts}(\mathcal{C}, \mathcal{C}), \circ, \text{id})$.
eg: $(\text{Cocts}(\text{Set}, \text{Set}), \circ, \text{id}) \simeq (\text{Set}, \times, \text{id})$
 $(\text{Cocts}(k\text{Vect}, k\text{Vect}), \circ, \text{id}) \simeq (k\text{Vect}, \otimes, k)$
- Small examples:
 - $\Delta_+ = (\text{finite ordinals } \underline{n} = \{0, \dots, n-1\}), \oplus = \text{ordinal sum.}$
 - Any monoid M becomes a discrete monoidal caty $(M, \cdot, 1)$
 - Any comon. monoid M becomes a one-object monoidal caty with $0 = \otimes = \circ$.

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We can also build new monoidal catys from existing ones.

Obvious: $\mathcal{V} \times \mathcal{W}$ monoidal if \mathcal{V}, \mathcal{W} are.

Less obvious: suppose \mathcal{V} is monoidal w/ coproducts and

\otimes preserves coprods in each variable (ie

$A \otimes (-), (-) \otimes A : \mathcal{V} \rightarrow \mathcal{V}$ pres. coprods).

- $\mathcal{V}^{\mathbb{N}}$ is monoidal via $(A \otimes B)_n = \sum_{n=m+k} A_m \otimes B_k$.
- $\mathcal{V}^{I \times I}$ is monoidal for any set I , via $(A \otimes B)_{ik} = \sum_{j \in I} A_{ij} \otimes B_{jk}$.
- $\mathcal{V} \times \mathcal{V}$ monoidal via $(A, C) \otimes (B, D) = (A \otimes B, A \otimes D + C \otimes B)$
Call this $\mathcal{V}[E]$ by analogy with ring of dual numbers.
- \mathcal{V} itself monoidal via $A \otimes' B = A + B + A \otimes B$ $I' = 0$
and $A \otimes'' B = A + B + AB + BA + ABA + BBA + \dots$ $I'' = 0$.

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A very important way of building mon. str. is via convolution.

DEFN Let \mathcal{V}, \mathcal{W} be monoidal catys, $F, G, H: \mathcal{V} \rightarrow \mathcal{W}$ ^{arbitrary} functors.

- A bilinear map $F, G \rightarrow H$ is a natural family of maps

$$F(X) \otimes_{\mathcal{W}} G(Y) \rightarrow H(X \otimes_{\mathcal{V}} Y) \quad \forall X, Y \in \mathcal{V}$$

- A 0-linear map $() \rightarrow H$ is a map $I_{\mathcal{W}} \rightarrow H(I_{\mathcal{V}})$

The Day convolution monoidal str \otimes_{conv} on $[\mathcal{V}, \mathcal{W}]$, when it exists, is characterized by the fact that

$$\frac{F, G \rightarrow H \quad \text{bilinear}}{F \otimes_{\text{conv}} G \rightarrow H \quad \text{nat xfm}} \qquad \frac{() \rightarrow H}{I_{\text{conv}} \rightarrow H}$$

The existence of \otimes_{conv} is guaranteed if \mathcal{V} small, \mathcal{W} cocomp, and \otimes in \mathcal{W} pres. colims in each variable. The explicit

formulae are then:

$$(F \otimes_{\text{conv}} G)(V) = \int^{W, X} \mathcal{V}(W \otimes X, V) \cdot F W \otimes G X$$

$$I(V) = \mathcal{V}(I, V) \cdot I$$

For example: • $\mathcal{V}^{\mathbb{N}}$ above is the conv. monoid structure from $(\mathbb{N}, \cdot, 1)$ to $(\mathcal{V}, \otimes, I)$.

- Convolution str. on $[\Delta_+^{\text{op}}, \text{Set}]$ gives join of augmented simplicial sets.

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DEFN A monoid in a monoidal caty \mathcal{V} is $M \in \mathcal{V}$ t/w
 $m: M \otimes M \rightarrow M$, $e: I \rightarrow M$ satisfying unit and assoc.
 axioms.

Examples: • In Set , Top , $[\Delta_+^{\text{op}}, \text{Set}]$: monoid, topological monoid,
 simplicial monoid.

- In Cat : strict monoidal caty ($\alpha = \lambda = \rho = \text{id}$)
- In $k\text{-Vect}$: k -algebra; in $\underline{\text{Ab}}$: ring
- In $[\mathcal{C}, \mathcal{C}]$ (or $\text{Coch}(\mathcal{C}, \mathcal{C})$): a monad (or cocts monad) on \mathcal{C} .
- In Δ_+ : $\underline{1} = \{0\}$ is a monoid, in fact, Δ_+ is the
 free monoidal caty on the monoid $\underline{1}$.
- In $\mathcal{V}^{\mathbb{N}}$: a graded monoid in \mathcal{V} : $(M_i)_{i \in \mathbb{N}}$ t/w
 $M_i \otimes M_j \rightarrow M_{i+j}$ $I \rightarrow M_0$ + axioms.

• In $\mathcal{V}^{I \times I}$: involves \mathcal{V} objects $(\mathcal{C}(i, j))_{i, j \in I}$ Hw

$$\mathcal{C}(j, k) \otimes \mathcal{C}(i, j) \rightarrow \mathcal{C}(i, k) \quad I \rightarrow \mathcal{C}(i, i).$$

+ axioms: next time, we'll see there are \mathcal{V} -categories with ob set I .

• In $\mathcal{V}[\varepsilon]$: (A, B) Hw $(A \otimes A, A \otimes B + B \otimes A) \rightarrow (A, B)$
 $(I, 0) \rightarrow (A, A)$

... monoid A Hw an A - A -bimodule B .

• In $(\mathcal{V}, \otimes', 0)$: \otimes' -monoid is a \otimes -semigroup (monoid without unit).

What about monoids in $[\mathcal{V}, \mathcal{W}]_{\text{cat}}$? Well, it's

$F: \mathcal{V} \rightarrow \mathcal{W}$ Hw

$$F, F \xrightarrow{m} F \quad () \xrightarrow{e} F$$

ie, maps $FX \otimes FY \xrightarrow{m_{xy}} F(X \otimes Y) \quad I \xrightarrow{e} FI \quad \otimes$,
 plus axioms. More generally:
 (lax)

DEFN A monoidal functor $F: \mathcal{V} \rightarrow \mathcal{W}$ between mon catys is
 a functor $F: \mathcal{V} \rightarrow \mathcal{W}$ Hw data \otimes plus axioms.

A monoidal functor is called strong if each \otimes is invertible.

Also have oplax mon. functors which involve maps $F(X \otimes Y) \rightarrow F(X) \otimes F(Y)$ and $F I \rightarrow I$ plus axioms.

There's a caty MonCat of monoidal caty + monoidal functors, and we can justify the importance of convolution by noting that

$$\frac{\mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{W} \quad \text{monoidal functors}}{\mathcal{U} \longrightarrow [\mathcal{V}, \mathcal{W}]_{\text{conv}} \quad \text{monoidal functors}}$$

2) BICATEGORIES

Just as monoids $\xrightarrow[\text{objects}]{\text{many}}$ categories; so monoidal caty $\xrightarrow[\text{objects}]{\text{many}}$ bicategories

DEFN A bicategory \mathcal{B} comprises:

- a set of objects $\text{ob}(\mathcal{B})$;
- for all $x, y \in \text{ob}(\mathcal{B})$, a hom-category $\mathcal{B}(x, y)$;
- for all $x, y, z \in \text{ob}(\mathcal{B})$, a composition functor $\otimes_y: \mathcal{B}(y, z) \times \mathcal{B}(x, y) \rightarrow \mathcal{B}(x, z)$

objects: $f: x \rightarrow y$ 1-cells
morphisms: $x \xrightarrow[\alpha]{f} y$ 2-cells

- for all $x \in \text{ob}(\mathcal{B})$, an identity 1-cell $I_x \in \mathcal{B}(x, x)$.

plus: nat isos:

$$\begin{array}{ccccc} & P & \xrightarrow{x} & N & \xrightarrow{y} & M \\ & \nearrow \omega & & \downarrow \alpha & & \searrow \tau \\ (& P & \xrightarrow{x} & N & \xrightarrow{y} & M &) \end{array}$$

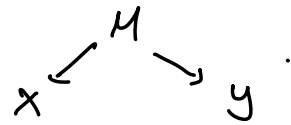
$$\alpha: (M \otimes_y N) \otimes_x P \longrightarrow M \otimes_y (N \otimes_x P) \quad \text{in } \mathcal{B}(u, z)$$

$$\lambda_u: I_y \otimes_y M \longrightarrow M \quad \rho_m: M \longrightarrow M \otimes_x I_x$$

satisfying 2 axioms. If $\alpha = \lambda = \rho = \text{id}$, call \mathcal{B} a 2-category.

Examples

- \mathbf{Cat} is a 2-category, where 2-cells are nat. transfs.
- \mathbf{MonCat} is a 2-category, where 2-cells are monoidal nat. transfs. Likewise, have $\mathbf{MonCat}_{\text{strong}}$, $\mathbf{MonCat}_{\text{oplax}}$.
- There's a bicat $\mathbf{Span}(\mathbf{Set})$, where objects are sets, and 1-cells $X \rightsquigarrow Y$ are spans



Composition is by pullback:



This is not a 2-cat;
only a bicategory.

More generally: $\mathbf{Span}(\mathcal{E})$ for any \mathcal{E} with pullbacks.

- If \mathcal{V} a mon caty w/ coproducts preserved by each $A \otimes (-)$, $(-) \otimes A$, get a bicat $\mathbf{Mat}(\mathcal{V})$ with:
 - obs: sets
 - 1-cells $I \rightarrow J$: objects of $\mathcal{V}^{I \times J}$
 - composition of $I \xrightarrow{M} J \xrightarrow{N} K$ is

$$(N \otimes_J M)_{ik} = \sum_{j \in J} N_{jk} \otimes M_{ij}.$$

Note: $\mathbf{Mat}(\mathcal{V})(I, I)$ is the $\mathcal{V}^{I \times I}$ we saw before.

In general, if \mathcal{B} any bicat, then $(\mathcal{B}(X, X) \otimes_X, I_X)$ is

monoidal category.

Note: $\text{Span}(\text{Set}) \sim \text{Mat}(\text{Set})$.

- There's a bicat $\underline{\text{Bim}}$ w/ rings as objects; 1-cells $M: R \rightsquigarrow S$ is a left R -right S -bimodule; composition is tensor product of bimodules.
- There's a 2-cat $\underline{\text{Rel}}$ with:
 - objects sets
 - 1-cell $R: X \rightarrow Y$ is a relation $R \subseteq X \times Y$
 - 2-cell $R \Rightarrow S: X \rightarrow Y$ is the assertion that $R \subseteq S$.
 - composition is relational composition.
- If \mathcal{C} is any cat, can see it as a 2-cat with only identity 2-cells;
- If \mathcal{V} is a monoidal cat, can see it as a one-object bicat.

We can now redevelop the preceding ideas for mon cats & for bicategories. For example:

- a monad in a bicat \mathcal{B} on an object $x \in \mathcal{B}$ is a monoid in $(\mathcal{B}(x, x), \otimes_x, I_x)$.
- a morphism of bicats $F: \mathcal{B} \rightarrow \mathcal{C}$ involves:
 - a map on objects: $\text{ob}(\mathcal{B}) \rightarrow \text{ob}(\mathcal{C})$;

- functors: $\mathcal{B}(x, y) \xrightarrow{\tau_{xy}} \mathcal{C}(F_x, F_y)$
- 2-cells: $F_{yz}(M) \otimes_{F_y} F_{xy}(N) \xrightarrow{m} F_{xz}(M \otimes_y N)$

$$I_{F_x} \xrightarrow{e} F(I_x)$$

... yielding a category Bicat of bicats and morphisms.
(call F a homomorphism if each m, e invertible).

- there's a notion of convolution bicat $[\mathcal{B}, \mathcal{C}]_{\text{conv}}$ such that

$$\begin{array}{l} \text{morphisms} \\ \text{morphisms} \end{array} \quad \frac{A \times B \rightarrow \mathcal{C}}{A \rightarrow [\mathcal{B}, \mathcal{C}]_{\text{conv}}}$$